A Modelica Implementation of the One-dimensional Shallow Water Equations on a Staggered Grid

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1 Shallow Water Equations

The one dimensional shallow water equations in conservative form are given by [?]:

$$\begin{aligned}\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - q = 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\frac{Q^2}{A} + gA\frac{\partial H}{\partial x} + c_f\frac{Q|Q|}{DA} - \frac{W\tau}{\rho_w} = 0\end{aligned} \tag{1}$$

with volume flow Q, water level above the plane of reference H, bed level H_b, total water depth $D = H - H_b$, channel cross section $A = A(H)$, channel width at the surface $W = W(H)$, lateral inflow per unit length q, wind-stress (parallel to the channel) τ, dimensionless bottom friction coefficient c_f, water density ρ_w, and gravitational acceleration g [?]. The friction coefficient should be calibrated to the model, but can be approximated $c_f \approx g/C^2$ where C is the Chézy coefficient.

2 Modelica Formulation

2.1 Staggered Grid

We define a Modelica model class containing an upstream port with a flow rate variable Q_u and a water level H_u, as well as a downstream port with flow rate variable Q_d and water level H_d. See Figure 1.

Inside the element, we discretize the shallow water equations on a staggered grid following [?]. Let n be the number of water level nodes, so that we have $n-1$ segments connecting the nodes. We associate a water level H_i, $i \in \{1, \ldots, n\}$, to every level node. In the middle of every segment and on the element boundaries, we define flow rates Q_j, $j \in \{1, \ldots, n+1\}$. On

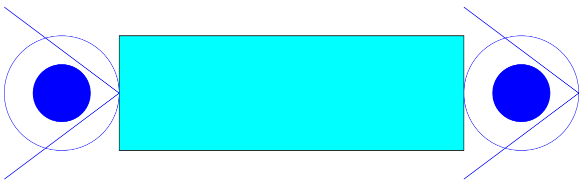

Figure 1: Model structure.

$$
\begin{array}{ccccc}
\multicolumn{2}{l}{\overset{\Delta x}{\longleftrightarrow}} & & & \\
Q_1 \; Q_2 & Q_3 & Q_4 & Q_5 \; Q_6 & \\
H_1 & H_2 & H_3 & H_4 & H_5 \\
D_1 & D_2 & D_3 & D_4 & D_5 \\
A_1 & A_2 & A_3 & A_4 & A_5 \\
\multicolumn{5}{r}{\longrightarrow x}
\end{array}
$$

Figure 2: Staggered grid with upstream and downstream boundaries.

the element boundaries, we set $Q_1 = Q_u$, $Q_{n+1} = -Q_d$[1], $H_1 = H_u$ and $H_n = H_d$. Figure 2 summarizes our grid configuration.

2.2 Discretized Equations

The discretized form of the shallow water equations 3 on the staggered grid of Figure 2 reads:

$$
\begin{aligned}
\frac{\partial A_i}{\partial t} - \frac{Q_i - Q_{i+1}}{\Delta x_i^q} &= q_i \qquad \forall i \in \{1, \ldots, n\} \qquad (2)\\
\frac{D_{i-1} + D_i}{2} \frac{A_{i-1} + A_i}{2} \left(\delta_I \left(\frac{\partial Q_i}{\partial t} + \delta_A \left(\frac{\Delta \frac{Q^2}{A}}{\Delta x} \right)_i \right) + \right. & \\
\left. \delta_P g \frac{A_{i-1} + A_i}{2} \frac{H_i - H_{i-1}}{\Delta x} - \frac{W_{i-1} + W_i}{2} \frac{\tau_i}{\rho_w} \right) + c_f Q_i |Q_i| &= 0 \qquad \forall i \in \{2, \ldots, n\}
\end{aligned}
$$

with

$$
\left(\frac{\Delta \frac{Q^2}{A}}{\Delta x} \right)_i = (1 - \delta_U) \frac{\frac{Q_{i+1}^2}{A_{i+1}} - \frac{Q_{i-1}^2}{A_{i-1}}}{\Delta x_{i-1}^q + \Delta x_i^q} + \delta_U \begin{cases} \frac{\frac{Q_i^2}{A_i} - \frac{Q_{i-1}^2}{A_{i-1}}}{\Delta x_{i-1}^q} & \text{if } Q_i \geq 0 \\ \frac{\frac{Q_{i+1}^2}{A_{i+1}} - \frac{Q_i^2}{A_i}}{\Delta x_i^q} & \text{otherwise} \end{cases}
$$

and

$$
\Delta x_i^q = \begin{cases} \Delta x / 2 & \text{if } i \in \{1, n\} \\ \Delta x & \text{otherwise} \end{cases}
$$

Terms are enabled and disabled using fixed boolean settings $\delta_I, \delta_A, \delta_P, \delta_U \in \{0, 1\}$.

It is to be noted that the bed level H_b does not appear explicitly in the momentum equation. As a result no artificial flow is induced by the bed level slope. So, if at rest initially, the system will remain at rest.

No time discretization is carried out at this point. Time discretization is performed by the Modelica compiler and/or any subsequent tooling according to the method that is most expedient for the application at hand. A typical choice for the time discretization would be the implicit Euler method. For a reduction of the computational burden on the rootfinding algorithm, one may choose a semi-implicit method following [?]. In any case, when choosing the time step size Δt, whether globally or adaptively, care must be taken to satisfy the CFL condition [?]:

$$
\frac{u \Delta t}{\Delta x} \leq C_{\max}
$$

[1] In Modelica, flow variables employ a sign convention: Flow variables have a positive value when flow enters the element, and a negative value when flow leaves the element.

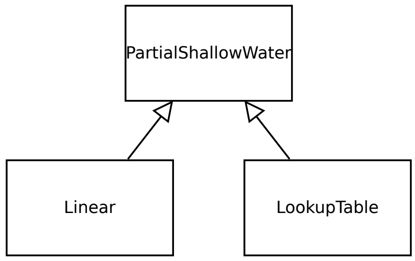

Figure 3: Class diagram of shallow water models.

with flow velocity u and $C_{\max}$ typically greater or equal than one.

The equations 2 on the grid of Figure 2 are implemented in the partial Modelica class `PartialShallowWater`. The relation between the cross area A and the water depth D is left unspecified. This relation is specified derived classes, as explained next.

2.3 Cross Sections

To complete the discretized equations 2, we need a relation between the cross area A and the water depth D. To this end, we supply two subclasses of the partial model `PartialShallowWater`: `Linear`, and `LookupTable`. The `Linear` model relates the cross section A to the total water depth D through a fixed width parameter W using the equation

$$A = WD.$$

The `LookupTable` model allows the modeler to insert a generic lookup table mapping the total water depth D to the cross section A.

The relations between the model classes are summarized in Figure 3.

2.4 Advection Across Element Boundaries

Suppose we connect two shallow water models together. We denote the variables of the upstream element using a superscript 1, and the variables of the downstream element using a superscript 2.

The connection equations are

$$\begin{aligned} H_d^1 &= H_u^2 \\ Q_d^1 + Q_u^2 &= 0 \end{aligned}$$

The mass balance equations left and right of the boundary separating the elements are

$$\begin{aligned} \frac{\partial A_n^1}{\partial t} - 2\frac{Q_n^1 + Q_d^1}{\Delta x^1} &= 0 \\ \frac{\partial A_1^2}{\partial t} - 2\frac{Q_u^2 - Q_2^2}{\Delta x^2} &= 0 \end{aligned}$$

Multiplying the first equation with $\Delta x^1/2$ and the second with $\Delta x^2/2$, and adding the resulting equations leads to

$$\frac{\partial V}{\partial t} - Q_n^1 + Q_2^2 = 0$$

where

$$V = \frac{\Delta x^1 A_n^1 + \Delta x^2 A_1^2}{2}$$

is a shorthand for the volume enclosed between the locations of Q_n^1 and Q_2^2. This shows that element connections are consistent with the continuity equation.

The mass balance equations can also be reorganized to give

$$\frac{1}{2}\frac{\partial U}{\partial t} + Q_u^2 = \frac{Q_n^1 + Q_2^2}{2}$$

where

$$U = \frac{A_n^1 \Delta x^1 - A_1^2 \Delta x^2}{2}$$

This shows that Q_u^2 is, in general, not equal to the average of Q_n^1 and Q_2^2. This would only hold in case the volumes on both sides of the boundary are equal, or if the system is in steady-state.

With regards to the advection, we have seen that

$$-Q_d^1 = Q_{n+1}^1 \neq Q_2^2$$

and

$$Q_u^2 = Q_1^2 \neq Q_n^1$$

In other words, element connections are not consistent with regards to the advection. However, in the limit as $\Delta x \to 0$, $-Q_d^1 = Q_u^2$ approaches Q_2^2 and Q_n^1 as long as Q is smooth. Consequently, for smooth solutions Q, the element connections are consistent with regards to the advection in the limit $\Delta x \to 0$.

An open research question is how to ensure consistency of the element connections with regards to the advection in Modelica for non-infinitesimal Δx.

2.5 Note on Initialization

The suggested way to initialize the model is by adding initialization equations for a steady state, i.e., by setting $\dot{Q}_i = 0$ for all $i \in \{2, \ldots, n\}$ at t_0. It may be necessary to provide the solver with an initial guess for water levels (or depths) to prevent it finding an initial state with negative water depths.

3 Test Cases

Ideas:

1. 1 element vs 2 elements: consistency Consistent if not using advection If advection is not taken into account,
2. Test convergence as $\Delta x \to 0$
3. Behaviour with and without inertia, advection, pressure, and upwinding
4. Optimization test case

5. Bottom slope comparison: Piecewise constant bottoms vs sloped bottoms Assume steady-state, neglect advection. Given a bottom slope $\partial H_b/\partial x < 0$ and a channel with uniform width W_0, a solution with uniform water depth D_0 would satisfy

$$Q_0^2 = -\frac{gD_0^3W_0^2}{c_f}\frac{\partial H_b}{\partial x} \tag{3}$$

Let $W_0 = 100\text{m}$, $D_0 = 10$ m, $\partial H_b/\partial x = -10^{-3}/g$, and $c_f = 10^{-2}$, so that $Q_0 = 10^3\text{m}^3/\text{s}$