Subgrid-scale model for Quasi-2D turbulence in shallow water

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ABSTRACT: Presently the simulation of larger-scale horizontal turbulent motions in shallow-water flows such as in harbours, lakes or rivers is feasible and it reduces the dependence on turbulence-closures applied in (unsteady) RANS. We call this Horizontal Large Eddy Simulation (HLES) i.e. simulating quasi 2D-turbulence with a dominant vertical vorticity component interacting with bed friction i.e. 3D turbulence.
This paper presents a subgrid-scale (SGS) model for the unresolved part of quasi 2D-turbulence in shallow water that is based on a *leaky* energy cascade i.e. with energy loss by bed friction. Rather than spatial filtering, a simple temporal recursive high-pass filter is applied on the resolved velocity field. This filter removes the slow or steady large-scale contributions to the strain rates that enter our SGS model. The SGS model has been successfully applied to simulating a shallow-water mixing layer, the eddies near groynes as well as the 3D stratified tidal flow near a harbour extension.

1 INTRODUCTION

Based on the hydrostatic-pressure assumption, our shallow-water solver Delft3D-Flow has been extensively used for simulating the depth-averaged as well as 3D flows in civil-engineering applications. This involves complicated (natural) lateral geometry, arbitrary bed levels, a multitude of open-boundary conditions including tides, atmospheric forcing, flooding and drying, density stratification etc.

For better assessment of navigation, structural stability, sediment transport, dredging operations, oil spills, algae blooms etc., there appears a growing interest in simulating more details of the flow. Examples are the temporal and spatial probability-density functions for horizontal velocity, bed-shear stress, mixtures of dissolved or suspended constituents, patchiness of oil spills or algae etc.. This information originates mostly from scales significantly larger than of 3D turbulence or the water depth but less than the scales introduced by tides. In this intermediate range of scales occurs 2D turbulence. In real flows, 2D turbulence is distorted by 3D effects and therefore it is designated as quasi-2D turbulence, consisting of eddies with a dominant vertical vorticity component.

The classical approach would be to follow the concepts of unsteady Reynolds-Averaged Navier-Stokes (URANS) computations and accordingly design a closure for quasi-2D turbulence. For the following reasons and experience, however, we object to this approach. The design of an adequate closure for quasi-2D turbulence is a complicated task due to the multiplicity of internal and external sources that induce or supply energy to quasi-2D turbulence (meteorology, topography, barotropic and baroclinic instabilities etc.). Inevitably, unknown coefficients appear that require calibration for which detailed observations are scarce. Finally, our experience with some attempts using an two-equation model for quasi-2D turbulence (Bijvelds et al., 1999) learns that a small grid size, of the order of the water depth, is still required to solve the unsteady RANS equations accurately.

In view of the latter computational requirement, we prefer simulating, rather than modeling, the evolution of quasi-2D turbulence in accordance with the physical laws of shallow-water flows. We aim at grid sizes of the order of the water depth so that 3D turbulence remains computationally unresolved and it is still modeled by the well-known closures. For obvious reasons we call this approach Horizontal Large Eddy Simulation (HLES). In this new approach, the grid size acts as a low-pass filter property rather than as accuracy criterion in RANS.

A particular type of turbulence closure is required for representing the action of the unresolved part of quasi-2D turbulence. The latter turbulence closure is called subgrid-scale (SGS) model as it depends on low-pass filtering of the physical flow by the grid size. Of course, compared to RANS, an SGS model is also a closure and the same arguments against its design and calibration could be applied as well. However, this paper demonstrates that its derivation can be performed without introducing new coefficients. Further, our SGS model is a simple algebraic expression yielding a marginal computational overhead. Finally, we show in this paper that the contribution of the SGS model can be of the order of the one by the 3D turbulence closure. In other words, by striving at simulating the horizontal turbulent currents down to sufficiently small grid sizes we reduce the dependency of the results on the additional SGS model.

The interpretation of HLES differs from the unsteady RANS. The latter is deterministic in its results that represent statistical means and variances. The results of HLES represent a particular realisation of quasi-2D turbulence, containing coherent structures. However, on-line processing of the flow variables of HLES also yields statistical properties but with more information e.g. about probability densities of the outcome of non-linear processes (erosion, bed friction etc.).

These are the arguments and motivations for striving at HLES and the principal step is the design of an adequate SGS model. The next section reviews the design criteria for such a model.

2 DESIGN CRITERIA FOR AN SGS MODEL IN SHALLOW FLOWS WITH COMPLICATED GEOMETRY

For reasons explained in this section, modern SGS models, founded on the concepts of the dynamic model (Germano et al., 1991), are not suitable for practical application of HLES. The available SGS models are dedicated to 3D-LES with simple geometries and mostly equidistant and (nearly) isotropic grids. In this section we present four design criteria for a suitable SGS model for shallow flows.

Due to physical properties of ideal 2D turbulence, most of the energy in the flow perturbations cascades to larger length and time scales (Lesieur, 1997) that are resolved adequately by the simulation, see fig. 1. Most of this so-called inverse energy cascade is due to the merging of 2D vortices with equal sign of their vertical vorticity vector (termed upscaling in fig. 1) as well as through the principle of enstrophy (square of vorticity) conservation (Fjørtøft, 1953).

A minor part of the energy in the supplied flow perturbations, however, is cascaded to smaller spatial and temporal scales. The latter process is mainly due to the interaction of vortices with opposite signs thereby creating small regions with intensified strain rates (McWilliams, 1990), called downscaling in fig. 1. This transfer of energy to smaller scales resembles the well-known energy cascade in 3D turbulence, although the upscaling flux or backscatter in 3D may be significant as well, see (Piomelli et al., 1991). Ideally, the rate-of-energy cascaded towards the computationally unresolved scales should be absorbed adequately by an SGS model. The latter is the first as well as the principal design criterion for any SGS model. In fig. 1 the downscaling flux of quasi-2D turbulence is defined as ε^{2D} and it acts as a source of 3D turbulence.

Three other conditions, however, invoke deviations from the previous idealized picture and these conditions outline an SGS model that differs from the available ones for 3D-LES.

The first deviating condition is due to a physical process typical of shallow-water flows, namely the work done by 2D turbulence against bed friction. Bed friction drains energy from the computationally resolved 2D turbulence motions. In general the shallower the water and the larger the kinetic energy of 2D turbulence, the more their energy is consumed by friction i.e. converted into 3D turbulence and the lesser their energy cascades, see e.g. (Uijttewaal & Booij, 2000). The direct conversion of 2D turbulence, at scales of 2D turbulence, to 3D turbulence may be called a *short-cut cascade* (fig. 1), a term borrowed from the destruction of atmospheric turbulence while penetrating plant vegetation. In other words, in case of shallow-water bed friction, the SGS model should drain less energy from 2D turbulence as it is already drained through the short-cut cascade.

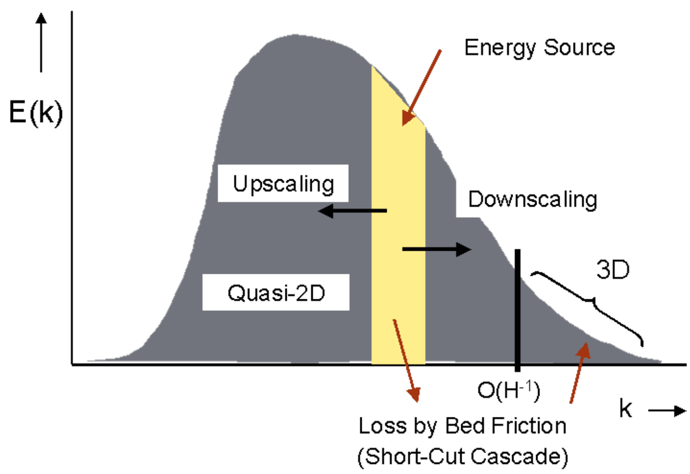

Figure 1. Spectral energy density E(k) vs. horizontal wave number magnitude k. Arrows indicate energy fluxes including the short-cut cascade of quasi-2D turbulence directly into 3D turbulence. The downscaling flux requires an SGS model if horizontal grid sizes exceed the water depth H.

The second deviating condition is the energy dissipation by the numerical solution method. For reasons of robustness and general applicability without time step limitations, our shallow-water solver is dissipative, although marginally (Uittenbogaard & Van Vossen, 2001). In many academic DNS and LES codes, energy conservation is guaranteed. Dedicated to HLES, the accompanying paper (Van Os & Uittenbogaard, 2003) proposes a variance-conserving advection scheme that avoids numerical dissipation. In any case, the SGS model should be sensitive to the dissipation imposed by the numerical method.

Finally, there is a third and practical deviating design constraint. It concerns the feasibility of the SGS model to complicated flow geometries, typical for civil-engineering flows. This requirement needs some introduction and problem analysis.

An essential deficit of the classical Smagorinsky SGS model is that it drains too much energy from the eddy motions when these are superimposed on shear flows such as wall-boundary layers, see e.g. (Germano et al., 1991). In Smagorinsky's closure the total strain rates of both the mean flow as well as turbulence determine the SGS eddy-viscosity: even in a non-turbulent shear flow Smagorinsky's SGS model would falsely respond by energy removal.

Instead, a proper SGS model should respond to turbulence strain rates only. To our opinion, therefore modern but academic SGS models such as the class of so-called dynamic models apply two spatial filters mostly through 2D Fourier transforms of the velocity field defined on equidistant grids. Essentially, the spatial filters act as high-pass filters that subtract the mean-flow contributions from the strain rates that enter the SGS model.

We desire to include the latter property also in our SGS model while keeping it feasible for complicated geometries, typical for civil-engineering flows. Our shallow-water solver is based on orthogonal curvilinear staggered grids allowing for variable grid sizes and including so-called thin dams or weirs by blocking just a single grid line. Consequently, the application of spatial filters is at least cumbersome and instead we prefer high-pass temporal filtering with the same objective of separating the quasi-2D turbulence from the gradual changes in tidal or wind-induced flow.

The four design criteria are summarized as follows:

1. adequate drainage of energy cascaded to the smallest resolved scales;
2. accounting for the energy lost through the short-cut cascade;
3. spatial low-pass filter properties adjusted to the properties of the numerical method;
4. feasible filter operation that removes mean flow contributions to the SGS model.

The next section presents the main derivation of the SGS model in accordance with these criteria.

3 SHALLOW-WATER SGS MODEL

The Reynolds-stress tensor $\underline{\underline{R}}^{(3D)}$ due to 3D turbulence is closed by the eddy viscosity $\nu^{(3D)}$ and reads:

$$\underline{\underline{R}}^{(3D)} - \tfrac{1}{3}\underline{\underline{I}}\, tr\left\{\underline{\underline{R}}^{(3D)}\right\} = 2\nu^{(3D)}\, \underline{\underline{S}}(<k_s) \qquad (3.1)$$

with unit-tensor $\underline{\underline{I}}$. The strain-rate tensor $\underline{\underline{S}}$ is based on the horizontal velocity vector $\underline{u}$:

$$\underline{\underline{S}} = \tfrac{1}{2}\left\{\nabla\underline{u} + (\nabla\underline{u})^T\right\} \qquad (3.2)$$

In (3.1), $\underline{\underline{S}}(<k_s)$ is based on the horizontal velocity vector $\underline{U}$. Vector $\underline{U}$ is the computationally resolved velocity. It is low-pass filtered at horizontal wave number magnitude k_s imposed by the grid size as well as the discretisation. This spatial filtering is noted by the $<k_s$ argument.

In (3.1), $\nu^{(3D)}$ is the eddy-viscosity due to 3D turbulence, assuming such a large Reynolds number that 3D turbulence occurs. The eddy-viscosity $\nu^{(3D)}$ is described by an adequate model for 3D turbulence e.g. the k-ε model in case of 3D computations, or in the depth-averaged mode the Elder formulation:

$$\nu^{(3D)} = \tfrac{1}{6}\kappa u_* H \qquad (3.3)$$

with bed-shear velocity u_*, water depth H and Von Kármán constant $\kappa \approx 0.4$.

Similarly, we close the Reynolds-stress tensor $\underline{\underline{R}}^{(SGS)}$ of quasi-2D turbulence by an eddy viscosity $\nu^{(SGS)}$. This closure reads

$$\underline{\underline{R}}^{(SGS)} - \tfrac{1}{3}\underline{\underline{I}}\, tr\left\{\underline{\underline{R}}^{(SGS)}\right\} = 2\nu^{(SGS)}\, \underline{\underline{S}}^* \qquad (3.4)$$

In (3.4) appears $\underline{\underline{S}}^*$ based on the high-pass temporal filtered $\underline{U}$ that is written as $\underline{u}^*$. Consequently, $\underline{\underline{S}}^*$ is spatially low-pass filter at k_s, as $\underline{U}$, but also temporarily high-pass filtered; section 4 defines the filter parameters.

Our closure (3.4) represents the spectral transport (or transfer) of kinetic energy of quasi-2D turbulence to the unresolved scales. This is the downscaling flux in fig. 1. The transferred energy originates from the computationally resolved quasi-2D turbulence. The temporal high-pass filter excludes the direct transfer of mean-flow energy to the SGS quasi-2D turbulence. The transfer of mean-flow energy to the resolved part of quasi-2D turbulence is simulated directly by the shallow-water solver.

A closure for $\nu^{(SGS)}$ is constructed by considering the depth-averaged balance of kinetic energy of SGS quasi-2D turbulence including the energy loss by bed friction (short-cut cascade). In absence of external forcing and for homogeneous and stationary SGS quasi-2D turbulence this balance yields:

$$\left\langle \underline{\underline{R}}^{(SGS)} : \underline{\underline{S}}^{*} \right\rangle - H^{-1} \left\langle \underline{u}^{(SGS)} \cdot \underline{\tau}_{bed} \right\rangle = 0 \qquad (3.5)$$

where the brackets indicate ensemble averaging. The first term of (3.5) represents the spectral transport of energy towards SGS kinetic energy of $\underline{u}^{(SGS)}$, the depth-averaged velocity, high-pass filter at horizontal wave number magnitude k_s. Using (3.5), the production of SGS quasi-2D turbulence is defined by

$$\left\langle \underline{\underline{R}}^{(SGS)} : \underline{\underline{S}}^{*} \right\rangle = \nu^{(SGS)} Q(k_s) \qquad (3.6)$$

where twice the sum of the resolved quasi-2D turbulence strain-rates below wave number k_s is expressed by

$$Q(k_s) = Q_{tot} - \int_{k_s}^{\infty} k^2 E(k)\, dk \qquad (3.7)$$

with Q_{tot} the constant total sum, E(k) the spectral-energy density of the kinetic energy of 2D turbulent motions at horizontal wave number magnitude k, see also (3.13). In (3.7), the integral covers the bandwidth of SGS quasi-2D turbulence and in this bandwidth it is assumed to be horizontally isotropic.

As usual in civil-engineering, the bed friction vector is quadratic in $\underline{u}$ and it is defined as

$$\underline{\tau}_{bed} = c_f |\underline{u}| \underline{u} \qquad (3.8)$$

with appropriate friction coefficient c_f and by definition $|\underline{\tau}_{bed}| = |u_*|^2$ holds with u_* appearing in (3.3). Splitting (3.8) into resolved and unresolved velocity vectors yields for the energy sink in (3.5):

$$H^{-1} \left\langle \underline{u}^{(SGS)} \cdot \underline{\tau}_{bed} \right\rangle = 2B \int_{k_s}^{\infty} E(k)\, dk \qquad (3.9)$$

where $B \approx \frac{3}{4} c_f |\underline{U}| H^{-1}$ holds for 2D isotropic turbulence. If the vertical profile of the horizontal velocity is logarithmic and the simulation is 3D then it can be shown that (3.9) represents the energy converted into 3D turbulence. Substitution of (3.7) into (3.6) and that result, together with (3.9), substituted into (3.5) yields, after differentiation with respect to the truncation wave number k_s:

$$Q \frac{d\nu^{(SGS)}}{dk_s} + \nu^{(SGS)} \frac{dQ}{dk_s} + 2BE(k_s) = 0 \qquad (3.10)$$

where $dQ/dk_s = k_s^2 E(k_s)$ holds, in virtue of (3.7).

A major effort of solving (3.10) for the SGS eddy- viscosity $\nu^{(SGS)}$ is expressing $\nu^{(SGS)}$ into the spectral energy density E. The first step is the assumption that the eddy-viscosity is proportional to the eddy-diffusivity $\Gamma^{(SGS)}$ through

$$\nu^{(SGS)} = \sigma_T \Gamma^{(SGS)} \qquad (3.11)$$

with σ_T the turbulence Prandtl-Schmidt number and typically $\sigma_T \approx 0.5\text{-}1.0$ holds.

The following expression for $\Gamma^{(SGS)}$ is derived in the Appendix:

$$\Gamma^{(SGS)}(k_s) = \frac{1}{2} \frac{I_\infty}{\left|u^{(SGS)}\right|} \int_{k_s}^{\infty} \frac{E(k)}{k} dk \qquad (3.12)$$

with $I_\infty \approx 0.844$ and where the rms of the SGS quasi-2D turbulence velocity has been defined by:

$$\left|u^{(SGS)}\right|^2 \equiv \frac{1}{2} \left\langle \underline{u}^{(SGS)} \cdot \underline{u}^{(SGS)} \right\rangle = \int_{k_s}^{\infty} E(k)\, dk \qquad (3.13)$$

Assuming a power-law

$$E(k) = \eta k^{-\alpha} \qquad (3.14)$$

for the spectral energy density with power $\alpha > 1$ and substitution of (3.14) in (3.13) as well as in (3.12) finally yields the derivative of $\nu^{(SGS)}$ as it appears in (3.10):

$$\frac{d\nu^{(SGS)}}{dk_s} = -\frac{1}{4} (\sigma_T I_\infty)^2 (1 - \alpha^{-2}) \frac{E(k_s)}{k_s \nu^{(SGS)}} \qquad (3.15)$$

The substitution of (3.15) into (3.10) yields an algebraic expression, quadratic in $\nu^{(SGS)}$, and its solution reads:

$$\nu^{(SGS)} = \frac{1}{k_s^2} \left(\sqrt{(\gamma \sigma_T)^2 \left(\underline{\underline{S}}^{*} : \underline{\underline{S}}^{*} \right) + B^2} - B \right) \qquad (3.16)$$

where $\gamma = \frac{1}{2} I_\infty \sqrt{1 - \alpha^{-2}}$ and $I_\infty \approx 0.844$ hold.

In section 5, (3.16) is compared to the depth-averaged eddy-viscosity (3.3) due to 3D turbulence. Further, section 5 investigates the damping role of the bed friction i.e. of the short-cut cascade on the SGS eddy-viscosity.

The final result (3.16), however, requires the specification of the truncating wave number k_s as well as the definition of the high-pass filter yielding $\underline{u}^{*}$ and

$\underline{\underline{S}}^*$. These filter operations are considered in the next section.

4 SPATIAL AND TEMPORAL FILTERING

Figure 2 presents the transfer function of the advection scheme currently employed in our shallow-water solver. Typically, at about 6 grid sizes Δx the wave amplitude is reduced significantly. Therefore, the truncation wave number is defined as

$$\frac{1}{k_s^2} = \frac{\Delta x\, \Delta y}{\left(\pi f_{lp}\right)^2} \tag{4.1}$$

and from numerical experiments we estimate $f_{\ell p} \approx 0.3$ such that $k_s \approx (\Delta x)^{-1}$ holds on a square grid. This estimate and (4.1) obey the third requirement for the SGS model, see Section 2. We refer to (Van Os & Uittenbogaard, 2003) for an advection scheme designed for energy conservation for which $f_{\ell p} = 1$ holds.

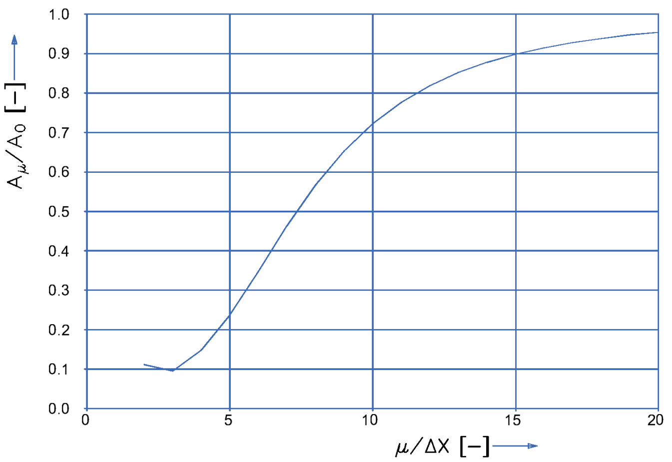

Figure 2. Reduction of velocity amplitude A_0 to A_μ at wave length μ by the mixed 2nd order upwind/central advection scheme (Stelling, 1984) of our shallow-water solver Delft3D-flow for μ/Δx grid points.

The fourth and final criterion of Section 2 is founded on the success of the so-called dynamic model and experiences reported in (Germano et al., 1991) who apply spatial filtering of the Fourier-transformed velocity field. Various authors (Domaradzki & Loh, 1999) and (Meneveau et al., 1996) propose spatial filters applicable in the physical domain. Nevertheless, all these appear at least cumbersome for our numerical grids and definitions of thin dams, weirs etc.. The essential aspect of such spatial filters appears to be the removal of the contribution of the mean strain rate to the SGS model.

The most simple and feasible alternative is temporal filtering of the velocity signals. We agree with (Meneveau et al., 1996) that then such filtering should be applied in a Lagrangian manner i.e. while following the same fluid parcel in space. Such a Lagrangian filter operator can be constructed straightforwardly using the general on-line transport routine in our shallow-water solver. The computational price to pay is an additional transport equation. Presently, we obtained good results (Kernkamp & Uittenbogaard, 2001) by using the following simple Eulerian recursive filter. For any scalar variable at time steps (n, n+1) it reads

$$\psi^* = \psi_{n+1} - \overline{\psi}^t_{n+1} \tag{4.2}$$

where the last term is the low-pass filtered property obtained through the recursive filter

$$\overline{\psi}^t_{n+1} = (1-a)\psi_{n+1} + a\overline{\psi}^t_n \tag{4.3}$$

with $\overline{\psi}^t_0 = 0$ and $a = \exp(-\Delta t / \tau)$. The time scale τ defined in this operator is specified by the user and should exceed the typical Eulerian time of the passage of eddies. On the other hand, τ should be significantly smaller than the time scales of external energy sources for the mean flow.

The previous definition of the temporal and spatial filters completes the design constraints of Section 2. The following section presents some analysis of the SGS model (3.16).

5 ANALYSIS OF THE SGS MODEL

A fortunate result of (3.16) is that it does not yield new calibration coefficients except the slope α of the energy spectrum (3.14). Typical slopes of 2D turbulence yield α=5/3 or 3 (Kraichnan, 1971) but the sensitivity of γ to α is then negligible.

In the following, the ratio between the SGS eddy-viscosity (3.16) and the depth-averaged viscosity (3.3) is estimated for the case of deep water (B=0). For square grids, this ratio reads

$$\frac{\nu^{(SGS)}}{\nu^{(3D)}} = \frac{\frac{1}{2}\gamma\sigma_T I_\infty \sqrt{1-\alpha^{-2}}}{\frac{1}{6}\kappa\left(\pi f_{lp}\right)^2} \frac{(\Delta x)^2}{HL} \frac{\left|u^*\right|}{u_*} \tag{4.1}$$

where we assumed $\sqrt{\underline{\underline{S}}^*:\underline{\underline{S}}^*} \approx \left|u^*\right| / L$. Consider $\left|u^*\right| / u_* = 1$, σ_T=0.7 and all other coefficients given previously then the SGS eddy-viscosity equals the depth-averaged one for $\Delta x \approx 0.5\sqrt{HL}$. If L=O(10)H then the grid size is about twice the water depth. Note that the ratio (4.3) increases proportionally to the horizontal grid area.

Finally, the reduction of the SGS eddy-viscosity by the bed-friction parameter B in (3.16) is investigated. First note that, for fixed strain rates, if B increases then $\nu^{(SGS)}$ decreases. The bed-friction parameter B increases when the quasi-2D turbulence is superimposed on a larger mean flow or a shallower flow. Both effects reduce $\nu^{(SGS)}$, as expected, and are due to the increased role of the short-cut energy cascade (fig. 1). Define

$$\nu^{SGS}(B>0) = f(z)\cdot \nu^{SGS}(B=0) \tag{4.2}$$

with $z = B/\gamma\,\sigma_T\sqrt{\underline{\underline{S}}^*:\underline{\underline{S}}^*}$ and the function

$$f(z) = \sqrt{1+z^2} - z \tag{4.3}$$

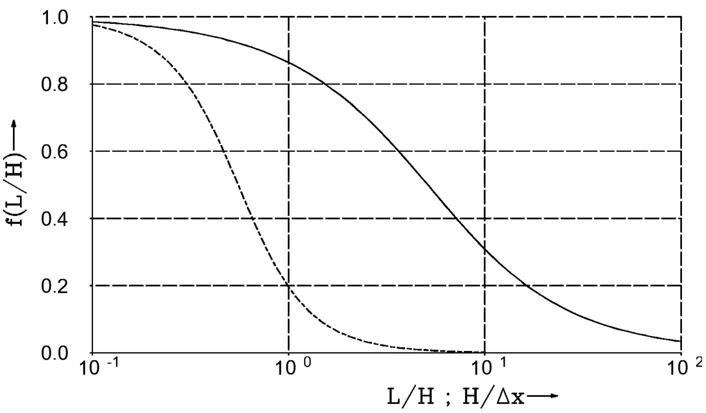

Figure 3. Example of reduction f (4.3) of $\nu^{(SGS)}$ through bed friction, depending on horizontal length scale (L) and water depth (H) or by grid size (dashed line, see Appendix).

Figure 3 presents and example for c_f=0.027, $\sqrt{\underline{\underline{S}}^*:\underline{\underline{S}}^*} = |u^*|/L$ with $|u^*|/U = 0.05$ and all other coefficients as applied before. This figure clearly shows the strong reduction in $\nu^{(SGS)}$ when the typical eddy size L is at least one order of magnitude larger than the water depth H.

6 RECIPES FOR A PROPER HLES

This section summarizes our experiences based on references given in section 7.

The SGS model (3.16) is a simple algebraic expression that, together with the temporal-filter operator (4.3), requires a marginal overhead in computational effort. Nevertheless, the computational effort is large. Firstly due to the requirement of resolving at least the most energetic vortex-vortex interactions on the grid. This requires a sufficiently fine horizontal grid, typically of the order of the water depth. Secondly, following the temporal evolution of the vortex-vortex interactions is also essential and therefore the time step should be small. This requirement is sometimes called *space-time consistency* that demands for the advection Courant number, with $\Delta\ell$ the grid size in flow direction:

$$C_U = \frac{|U|\Delta t}{\Delta\ell} \le 1 \tag{6.1}$$

In absence of any internal or external friction, the flow solver should at least approximate well but preferably strictly obey the principle of energy conservation and preferably also the conservation of enstrophy (square of vorticity). Both conditions are fulfilled by the complicated scheme of Arakawa & Lamb (1981). For our shallow-water solver, however, we must impose the condition

$$C_{BT} = \left\{(\Delta x)^{-2} + (\Delta y)^{-2}\right\}^{\frac{1}{2}} \sqrt{gH}\Delta t \le 2\sqrt{2} \tag{6.2}$$

on the barotropic Courant number for approximating energy and enstrophy conservation, for details see (Uittenbogaard & Van Vossen, 2001).

Flow instabilities such as due to curved mean-flow profiles e.g. free-shear layers are the most prominent internal sources of generating quasi-2D turbulence. Also the vorticity generated locally by the friction of lateral walls (groynes, headlands, quay walls etc.) can be advected into the main flow volume (Clercx et al., 1999). Therefore friction by lateral walls is included in our shallow-water solver with a user-defined roughness length.

Typical for shallow-flow problems is that the physical flow domain is truncated by the grid. Open boundary conditions then should introduce the properties of the upstream current including quasi-2D turbulence. To that purpose Kernkamp & Uittenbogaard (2001) superimpose so-called kinematic turbulence to the standard inflow-boundary conditions.

The next section concludes this paper and it presents references to examples on which the previous recipes are founded.

7 CONCLUSIONS

The SGS eddy-viscosity model (3.16) is designed for shallow-flow applications and it fulfills our four design criteria, see section 2. This model does not yield new calibration coefficients. The model hardly increases the computational effort, as it is a simple algebraic expression combined with the simple temporal filter (4.3). Our SGS model accounts for the additional energy loss of quasi-2D turbulence by bed friction (fig. 1 and 3). This model is combined with available closures for 3D turbulence, e.g. the k-ε model. For grid-sizes of about twice the water depth the SGS eddy-viscosity has the same order of magnitude as the depth-averaged eddy-viscosity (3.3) due to 3D turbulence.

This SGS model has been implemented in our shallow water solver Delft3D-Flow and it is applied in depth-averaged as well as in 3D simulations for e.g. stratified flows.

Kernkamp & Uittenbogaard (2001) report a good comparison against the shallow-water mixing layer experiments in (Uijttewaal & Booij, 2000). Similar favorable results are reported in (Schijndel & Jagers, 2003) as well as the comparison of 3D simulations with extensive field observations (Bijlsma et al., 2003).

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APPENDIX

This appendix is devoted to deriving (3.12) that closes the algebraic equation (3.10) for the SGS eddy-viscosity $\nu^{(SGS)}$.

From the transport equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\underline{u}\,\phi) = 0 \qquad \text{(A.1)}$$

of any conserved scalar ϕ follows

$$\phi'(\underline{x},t) = \phi'\left[\underline{Y}(0;\underline{x},t),0\right] - \int_0^t \left\{\underline{u}'.\nabla\langle\phi\rangle\right\}\Big|_{(\underline{Y},\tau)} d\tau \qquad \text{(A.2)}$$

for its turbulence perturbation ϕ'. As a function of time τ appears in (A.2) path $\underline{Y}$ of a fluid parcel that ends in the Eulerian position $\underline{x}$ at time t. The time-space position ($\underline{x}$,t) is added as label after the semi-colon in the argument list of $\underline{Y}$. The brackets <..> imply ensemble averaging. From (A.2) follows the formal closure for the scalar flux $\langle \underline{u}'\phi'\rangle$ while neglecting higer-order derivatives of $\langle\phi\rangle$:

$$\langle \underline{u}'\phi'\rangle\big|_{(\underline{x},t)} = -\left\{\int_0^t \left\langle \underline{u}'(\underline{x},t)\underline{u}'\left[\underline{Y}(\tau;\underline{x},t)\right]\right\rangle d\tau\right\}\cdot\nabla\langle\phi(\underline{x},t)\rangle \qquad \text{(A.3)}$$

In view of the integrand in (A.3) the Lagrangian correlation tensor is defined by

$$\underline{\underline{R}}^{(L)}(\tau;\underline{x},t) = \left\langle \underline{u}'(\underline{x},t)\underline{u}'\left[\underline{Y}(\tau;\underline{x},t)\right]\right\rangle \qquad \text{(A.4)}$$

and the eddy-diffusivity tensor for (A.3) as:

$$\underline{\underline{\Gamma}} = \int_0^t \underline{\underline{R}}^{(L)}(\tau;\underline{x},t)\,d\tau \qquad \text{(A.5)}$$

The principal task now is expressing (A.5) for horizontally homogeneous 2D turbulence into the spectral energy density E(k), see (3.13). This is achieved by expressing the velocity vector into Fourier-Stieltjes integrals (Batchelor, 1953) as:

$$\underline{u}'(\underline{Y},\tau) = \int \exp(i\underline{k}\cdot\underline{Y})\,d\underline{Z}_u(\underline{k},\tau)$$

$$\underline{u}'(\langle\underline{Y}\rangle,t) = \int \exp(i\underline{l}\cdot\langle\underline{Y}\rangle)\,d\underline{Z}_u(\underline{l},t)$$

where $\underline{k}$ and $\underline{\ell}$ are horizontal wave number vectors. Substitution of these integrals in (A.5) and neglecting correlations between $\underline{Y}'$ and the Fourier coefficients $\underline{Z}_u$ yields:

$$\underline{\underline{R}}^{(L)}(\tau;\underline{x},t) = \int_{\underline{k}} \left\langle \exp(i\underline{k}\cdot\underline{Y}') \right\rangle \int_{\underline{l}} \left\langle d\underline{Z}_u(\underline{k},\tau) d\underline{Z}_u^{cc}(\underline{l},t) \right\rangle \tag{A.6}$$

The temporal spectral-energy density tensor is defined as

$$\underline{\underline{E}}(\underline{k},t-\tau)d\underline{k} = \int_{\underline{l}} \left\langle d\underline{Z}_u(\underline{k},\tau) d\underline{Z}_u^{cc}(\underline{l},t) \right\rangle$$

and Kraichnan (1959) derives the splitting

$$\underline{\underline{E}}(\underline{k},t-\tau) = \underline{\underline{E}}(\underline{k},0)\, h(\underline{k},t-\tau) \tag{A.7}$$

with $\underline{\underline{E}}(\underline{k},0)$ the spectral-energy density tensor

and $h(\underline{k},T)$ the transfer function decaying for increasing time difference $|T|$. Kraichnan (1959, eq. 5.3 for $\nu=0$) derives:

$$h(\underline{k},T) = \theta^{-1} J_1(2\theta)\ ;\ \theta = T\left(\underline{k}\cdot\langle\underline{u}'\underline{u}'\rangle\cdot\underline{k}\right)^{\frac{1}{2}} \tag{A.8}$$

with J_1 the Bessel function of the first kind. For small T the transfer function decays as $\exp\left(-\frac{1}{2}\theta^2\right)$. What remains in (A.6) is the first integrand that can be expressed into

$$\left\langle \exp(i\underline{k}\cdot\underline{Y}') \right\rangle = \exp\left(-\tfrac{1}{2}\underline{k}\cdot\langle\underline{Y}'\underline{Y}'\rangle\cdot\underline{k}\right) \tag{A.9}$$

using cumulant expansion while neglecting higher-order moments. For short arrival time intervals ($t-\tau$) $\langle\underline{Y}'\underline{Y}'\rangle = \langle\underline{u}'\underline{u}'\rangle(t-\tau)^2$ holds so that after all substitutions (A.5) reads:

$$\underline{\underline{\Gamma}}(t) = \int_{\underline{k}} \left\{ \int_{\tau=0}^{t} W(\theta)\, d\tau \right\} \underline{\underline{E}}(\underline{k},0)\, d\underline{k} \tag{A.10}$$

with the integrand

$$W(\theta) = \theta^{-1} J_1(\theta) \exp\left(-\tfrac{1}{2}\theta^2\right) \tag{A.11}$$

and θ defined by (A.8). Numerical integration yields:

$$I_\infty = \int_0^\infty W(\theta)\, d\theta \approx 0.844 \tag{A.12}$$

All this substituted into (A.6) and (A.5) yields:

$$\lim_{t\to\infty} \underline{\underline{\Gamma}}(t) = I_\infty \int_{k_s}^{\infty} \frac{\underline{\underline{E}}(\underline{k})}{\left(\underline{k}\cdot\langle\underline{u}'\underline{u}'\rangle\cdot\underline{k}\right)^{\frac{1}{2}}} d\underline{k} \tag{A.13}$$

The expression (A.13) holds for the eddy-diffusivity tensor due to horizontally homogeneous 2D turbulence. For isotropic 2D turbulence (A.13) simplifies to:

$$\lim_{t\to\infty} \Gamma(t) = \frac{1}{2} \frac{I_\infty}{|u'|} \int_{k_s}^{\infty} \frac{E(\underline{k})}{k} d\underline{k} \tag{A.14}$$

In the main text (A.14) is used as (3.12) i.e. with u' of the total turbulence velocity replaced by $u^{(SGS)}$ and by considering the resolved velocity field as mean flow. This ends the derivation for (3.12).

From the previous derivation, the following instructive conclusions can be drawn. Substitution of the power law (3.14) in (A.14) yields

$$\lim_{t\to\infty} \Gamma(t) = \frac{\alpha-1}{\alpha} I_\infty k_s^{-1} |u'| \tag{A.15}$$

The expression (A.15) resembles the classical mixing-length closure but here the mixing length is replaced by the grid size, see also (4.1).

In 3D simulations, the SGS-closure is combined with a 3D turbulence model and then (3.16) could be replaced by:

$$B = \nu^{(3D)} k_s^2 \tag{A.16}$$

with the depth averaged $\nu^{(3D)}$ defined by (3.3). Assuming $\sqrt{\underline{\underline{S}}^*:\underline{\underline{S}}^*} \approx |u^*|/L$, $|u^*|/u_* = 1$, $\sigma_T=0.7$ and all other coefficients given previously then for $L=10H$ the damping of the SGS eddy-viscosity is represented by the dashed line in fig. 3 showing a vanishing SGS eddy-viscosity for $\Delta x \le H$, as expected.

Finally a brief remark on the modeled spectral-energy fluxes (fig. 1). The work done by the SGS stresses against the total flow, i.e. mean-flow as well as the resolved part of quasi-2D turbulence, reads:

$$\left\langle \underline{\underline{R}}^{(SGS)} : \underline{\underline{S}} \right\rangle = 2\nu^{(SGS)} \left\{ \left\langle \underline{\underline{S}}^* : \underline{\underline{S}}^* \right\rangle + \left\langle \underline{\underline{L}} : \underline{\underline{S}}^* \right\rangle \right\} \tag{A.16}$$

where $\underline{\underline{L}} = \underline{\underline{S}} - \underline{\underline{S}}^*$ is the strain rate of the slow and most likely mean flow. Note that the last term of (A.16) may be negative with a magnitude exceeding the positive $\left\langle \underline{\underline{S}}^* : \underline{\underline{S}}^* \right\rangle$. In that case backscatter of SGS quasi-2D turbulence to the larger-scale flow occurs.