CATEGORIZATION OF TRAPEZOIDAL OPEN CHANNELS BASED ON FLOW CONDITIONS FOR THE CHOICE OF SIMPLE MODELS

Klaudia Horváth[1]
Mailing address
K.Horvath@tue.nl

Bart, P.M., van Esch, Jorn, H., Baayen, I. Pothof
Mailing address
B.P.M.v.Esch@tue.nl, Jorn.Baayen@deltares.nl, Ivo.Pothof@deltares.nl

Many applications in water management rely on keeping the water levels of an open water channel within given bounds, e.g. irrigation canals, drainage systems, and hydropower systems. These are all open water channels where the water level is influenced by several known and unknown factors like precipitation, operation of structures, etc.
Water levels can be efficiently controlled by model predictive control (MPC). In MPC the optimization algorithms give advice at every time step based on the current state of the system as well as on the expected future state. These algorithms need a model to predict the response of the system to the control inputs. In most cases, the need to guarantee convexity of the optimization problem leads to the requirement that these models should be linear. To date, several such linear models are available in literature, which are suited for control purposes. However, the choice between these models is not straightforward. In this work, we extend a categorization of open channels, based on which the choice of a simple model can be advised.

Key Words:
Control, modelling, propagation, linear, channel

Catégorisation des canaux ouverts trapézoïdaux en fonction des conditions d'écoulement pour le choix des modèles simples

L'objectif de gestion de la majorité des systèmes hydrauliques à surface libre, comme les canaux d'irrigation, les systèmes de drainage ou les systèmes hydro-électriques consiste à maintenir leurs niveaux autour de consignes et dans un intervalle de fonctionnement. Le respect de cet objectif requiert la conception de lois de commande. Cependant, plusieurs facteurs extérieurs connus ou inconnus, parmi lesquels il est possible de citer de manière non exhaustive la pluie et les incertitudes sur le fonctionnement des structures hydrauliques, peuvent perturber l'efficacité de ces lois de commande. Pour y faire face, les approches de régulation prédictive de type MPC (Model Predictive Control) ont été conçues. Elles présentent l'avantage de mettre en œuvre des algorithmes d'optimisation tenant compte à chaque instant de l'état courant du système et de la prédiction de ses états futurs. La prédiction des états futurs requiert un modèle du système afin de connaître la réponse du système à des entrées contrôlées. Ces modèles doivent être, dans la majorité des cas, linéaires afin de pouvoir garantir également la convexité du problème d'optimisation. Plusieurs modèles linéaires dédiés à la conception de lois de commande ont été proposés dans la littérature. Cependant, la sélection d'un de ces modèles linéaires pour un cas d'étude donné reste encore complexe. Ainsi, dans cet article, nous proposons des critères de choix permettant à partir des caractéristiques des systèmes hydrauliques à surface libre considéré et de son mode de fonctionnement d'orienter le choix du type de modèle linéaire.

Mots-clefs:
régulation, modélisation, propagation, linéaire, canaux

1. INTRODUCTION

Automatic control of water levels can enhance the saving of energy and water. Several automatic controller techniques, like model predictive control, require a simple internal model. In most cases the requirement is a linear model. Several simple models exist for control purposes [11, 8, 13] and their applicability has been discussed [12, 7, 6], but the question how to choose between these models remains unanswered.

[1] Corresponding author

There might not be one perfect model that suits all cases. Instead, it is more likely that the applicability of a model rather depends on the characteristics of a particular open water channel. The first step is thus to categorize general open channel flows. In literature, some of these categorizations are found, but only for special cases like normal flows [Baume and Sau, 1997].

In this work the existing categorization is extended for general open channel flows, and the flow behaviour of a number of test cases is analysed. Based on this categorization, an advice for the choice of the simple model can be given.

2. METHODOLOGY

Several categorizations of open water channels exist, each valid for a specific class of channel flows. In this section the existing categorizations are reviewed and a new, generalized categorization is proposed.

2.1. Existing categorizations

2.1.1. Categorization of normal channel flows

Fluctuations in discharge and water depth are mainly determined by two physical phenomena: wave propagation (perturbation) and mass transport (long waves) [Baume et al., 1998]. From the linearised Saint-Venant equations two dimensionless numbers are derived to categorize open channel flows based on normal flow conditions:

$$\chi = \frac{S_b L}{y} \tag{1}$$

and

$$\eta = \frac{\chi}{Fr(1-Fr)} \tag{2}$$

where Fr is the Froude number, which can be obtained as

$$Fr^2 = \frac{Q^2 T}{gA^3}. \tag{3}$$

In the above equations S_b is the bottom slope, L the length of the channel, y the reference depth, Q the reference discharge, A the reference cross section, and T the reference top width. As this categorization is valid for normal flow conditions, y denotes the normal depth.

Parameter χ is called the dimensionless length. It characterizes the transport of mass, i.e. the propagation of a discharge wave in the channel. The parameter is obtained as the ratio of the elevation of the bottom along the channel and the normal water depth [Baume et al., 1998]. For small values of χ, the water depth at the discharge of the channel responds as a first order system to an increase of upstream discharge. For high values of χ, the mass of water will transport fast and the downstream water level will respond like a second order system with delay. Thus, three categories for χ are distinguished: first order, second order, and second order with delay (see Figure 1 and Table 1).

Parameter η characterises the propagation of a downstream perturbation. It describes how far a wave from the downstream end can reach upstream. It is derived from the Froude number. In order to see more what influences η, the equation is simplified for a rectangular channel:

$$\eta = \frac{\chi}{Fr(1-Fr)} = \frac{S_b L}{yv(1-v/c)}. \tag{4}$$

If the wave coming from downstream is rapidly damped out, then $\eta > 3$. If η is smaller, the wave from downstream can travel up and down in the system several times.

Note that the dimensionless length, χ, only depends on the reference water level, while η, characterising the propagation of a perturbation, depends on both the reference discharge and the water level. Equation 4 has real solutions for $\eta \geq 4\chi$, therefore only the upper half Figure 1 is used

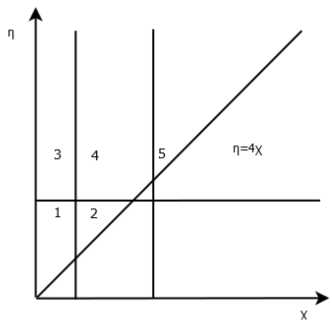

Figure 1: Canal categories, from [Baume and Sau, 1997]

	$0<\eta<3$	$\eta>3$
$0<\chi<0.6$	wave, first order (1)	damped wave, first order (3)
$0.6<\chi<1.35$	wave, second order (2)	damped wave, second order (4)
$\chi>1.35$		damped wave, second order with delay (5)

Table 1: Reach characteristics

2.1.2. Dimensionless length for trapezoidal canals with normal flow

In [Litrico and Georges, 1999] the Hayami equation is used to catergorize canals based on mass transport. The equation follows from a first order linearisation of the diffusive wave equation. Therefore the obtained categories are the same as the categories based on the χ parameter in [Baume et al., 1998]. The non-linear diffusive wave equation is:

$$\frac{\partial Q}{\partial t} + C(Q,z,x)\frac{\partial Q}{\partial x} - D(Q,z,x)\frac{\partial^2 Q}{\partial x^2} = 0 \tag{5}$$

where $C(Q,z,x)$ is the celerity of the diffusive wave and $D(Q,z,x)$ is the diffusivity. These are given by:

$$C(Q,z,x) = \frac{1}{T^2(\partial S_f / \partial Q}\left[\frac{\partial T}{\partial x} - \frac{\partial (TS_f)}{\partial z}\right] \tag{6}$$

$$D(Q,z,x) = \frac{1}{T(\partial S_f / \partial Q)}. \tag{7}$$

By computing C_0 and D_0 around a reference discharge (Q), the linear Hayami equation can be obtained:

$$\frac{\partial q}{\partial t} + C_0\frac{\partial q}{\partial x} - D_0\frac{\partial^2 q}{\partial x^2} = 0. \tag{8}$$

In [Litrico and Georges, 1999] these coefficients are computed using the following assumptions:

- rectangular channel
- $B \gg y$, therefore $R \approx y$ where *B* is the bottom width.
- diffusive wave, inertia terms are negligible ($\frac{\partial Q}{\partial t} + \frac{\partial Q^2 / A}{\partial X}$)
- normal flow conditions.

The resulting coefficients are:

$$D_0 = \frac{Q}{2TS_b} \tag{9}$$

$$C_0 = \frac{5}{3}\frac{Q}{Ty}. \tag{10}$$

In fact, the condition of uniform flow was only used to compute D_0. The computations are detailed in the appendix. Using the moment matching method, it is possible to develop three different categories. The development of the third category (second order with delay) is described in [Malaterre, 1994]. The condition for categorizing channels based on mass transport (dimensionless length) based on a coefficient CF :

$$CF_{unif} = \frac{L}{2}\frac{C}{D} = \frac{5}{3}\frac{LS_b}{y} \tag{11}$$

The subscript $unif$ refers to uniform flow. The channels are modelled with first order model, second order or second order with delay. This coefficient is the multiple of the χ coefficient from [Baume et al., 1998], and the boundary of the categories is the same. Therefore the dimensionless length can be computed as:

$$\chi = \frac{3}{5}CF. \tag{12}$$

The categories for length are developed in the following way. First the diffusive wave equation is deduced and linearized. From this linearized equation an analytical expression between the upstream and downstream discharges, the Hayami transfer function, is developed. During the development semi-infinite channel was assumed - this is valid for long channels. (Later in the article it is used for not too long canals). From the Hayami transfer function the different momenta are calculated. Depending on the parameter χ different number of momenta can be matched with first / second order functions. If the reach is small, the second order function is unstable, therefore a first order with delay function should be matched.

2.1.3. Formulas for dimensionless length for trapezoidal canals

In [Brotons, 2004] rectangular and trapezoidal example canals are categorized for mass transport based on the above described categoeisation from [Litrico and Georges, 1999]. Still normal flow is used. During the development of the formulas the following assumptions were used:

- the bottom slope is not zero
- $\frac{\partial P}{\partial y} \approx 0$, where P is the wet perimeter. This assumption is valid for wide channels.

The final formulas are shown in Table 2. These formulas are applied to two example canals: a rectangular and a trapezoidal one (Table 5), assuming normal flow. The resulting categorisation is shown in Figure 2 and Figure 3. It can be seen that in both cases the bigger the discharge is, the “ longer" the canal is (dimensionless length), in other words the higher category it is (second order with delay). This is also valid of course for the physical length of the canal: the bigger the physical length is, the bigger the dimensionless length.

B (m)	S(-)	z (-)	n (s/m1/3)
4	0.0005	0	0.014
4	0.0005	0.577	0.014

Table 5. Parameters of the canals used in [Brotons, 2004]

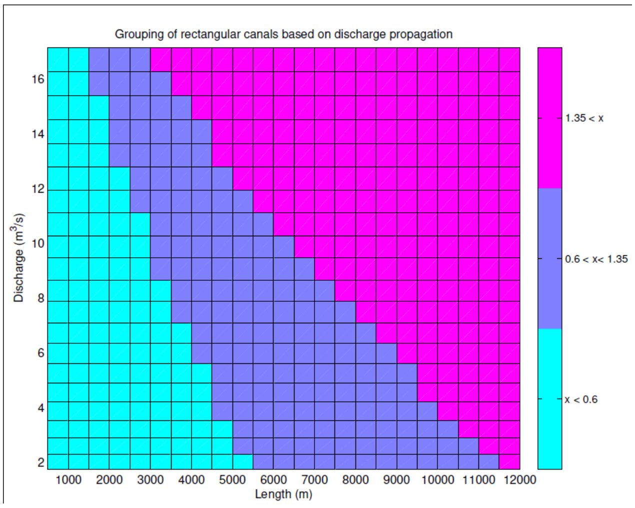

Figure 2: Groups based on mass transport using a rectangular test canal [Brotons, 2004]

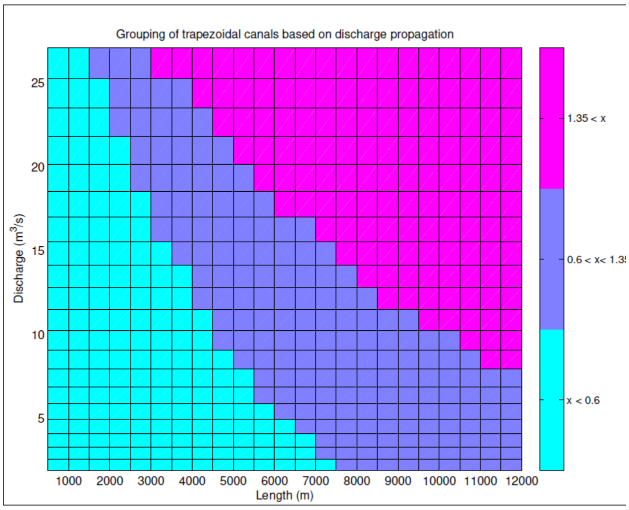

Figure 3: Groups based on mass transport using a trapezoidal test canal [Brotons, 2004]

2.1.4. Formulas for wave behaviour

In [van Overloop et al., 2010] a model is proposed for trapezoidal canals affected by resonance. These are canals where the waves can travel both upstream and downstream. The wave modelling contains the damping factor, ζ :

$$\zeta = \frac{\sqrt{2g}QL}{4C^2RBy^{3/2}} \tag{13}$$

where C is the Chézy coefficient and R is the hydraulic radius. For $\zeta < 1$ the presence of the waves dominates. This formula is valid for trapezoidal sections and not only during normal flow. This formula is valid for general cases. Assuming uniform flow conditions (4) can be derived from it (See Appendix).

2.1.5. Summing up existing categorization

The base of the categorization for mass transport and for wave reflection behaviour is presented in [Baume et al., 1998]. However, the development is assumed normal flow and formulas are given only for rectangular case. In [Litrico and Georges, 1999] the same categories are obtained for mass transport, using the same assumption. In [Brotons, 2004] the mass transport categorization presented in [Litrico and Georges, 1999] is analyzed and extended for wide trapezoidal canals with non-zero bottom slope. In [van Overloop et al., 2010] canals with resonance are treated, and a simple model is developed. These formulas for wave behaviour can be further used for categorization.

2.2. The proposed, new categorization

In this paper we propose a categorization valid for rectangular and trapezoidal channels, without restrictions of bottom slope or side slope, in any flow conditions. In [Baume and Sau, 1997] the reference condition is chosen as normal flow. The fact of normal flow is used to obtain a reference water depth. As the drainage canals are under backwater effect most of the time, we obtain a reference depth (that is not necessarily the normal depth) for the categorization. The categorization is based on the dimensionless length and wave propagation just as in [Baume et al., 1998]. The same variables are going to be used, but different formulas will be given, so that they could be computed for any condition. First, the categorization based on dimensionless length is explained, then the categorization based on wave characteristics, and then the final formulas are proposed.

2.2.1. Proposed categorization based on dimensionless length

In order to obtain formulas for dimensionless length, the similar principle as in [Litrico and Georges, 1999] used the linearized Hayami equation derived through moment matching method. However, when the linear coefficients are calculated, then neither the assumption of normal flow, nor any assumption of the side slope of the channel is used. Thus, from Equations 6 and 7, the following proposed (subscript $_p$) coefficients are obtained:

$$C_p = \frac{Q}{2T^2 S_f}\left[2z\frac{S_b - S_f}{1-Fr^2} - 2zS_f - TS_f\left(\frac{-10/3(B+2zy)+8/3R\sqrt{1+z^2}}{A}\right)\right] \quad (14)$$

and

$$D_p = \frac{Q}{2TS_f}. \quad (15)$$

In these formulas y is the water depth (around which the diffusive wave equation is linearised), z is the side slope of a trapezoidal channel and S_f is the friction slope from the Manning's equation. In case of normal flow, all the channel has the same normal depth, thus the choice is trivial. However, for backwater profile a representative water level should be chosen. As a first approach, it is chosen as the water depth in the middle of the backwater part, similarly as in [Litrico and Fromion, 2004b]. From the proposed coefficients, the proposed χ for the categories can be obtained as:

$$\chi_p = \frac{3L}{10}\frac{C_p}{D_p}. \quad (16)$$

The different ways to obtain the coefficients C and D are summarized in Table 2. It can be seen that D has the same formula in all cases. Coefficient C is different for the three cases. The formula by Brotons [2004] is the extension of [Litrico and Georges, 1999]: there is an additional term with side slope, containing the correction for trapezoidal channels. Note that for rectangular canals, with the assumption of $B \geq y$ the formula reduces to the one presented in [Litrico and Georges, 1999].

[Brotons, 2004]	$C_{Brotons} = \frac{5Q}{3A} - \frac{2Qz}{T^2}$	$D_B = \frac{Q}{2TS_f}$
[Litrico and Georges, 1999]	$C_{Lit} = \frac{5Q}{3A}$	$D_L = \frac{Q}{2TS_b}$
Proposed	$C_p = \frac{Q}{2T^2 S_f}\left[2z\frac{S_b - S_f}{1-Fr^2} - 2zS_f - TS_f\left(\frac{-10/3(B+2zy)+8/3R\sqrt{1+z^2}}{A}\right)\right]$	$D_p = \frac{Q}{2TS_f}$

Table 2: Formulas to calculate the coefficients for the linearised diffusive wave equation

2.2.2. Proposed categorization based on wave behaviour

To determine if there is a reflected wave travelling upstream the simple model from [van Overloop et al., 2010] is used. According to that work, the wave is present, if there is damping, $\zeta < 1$. This condition can be used in the categorization to see if upstream direction wave propagation is present. To make the limits coincide with the existing categorization, the wave propagation variable can be proposed as:

$$\eta_p = 3\zeta = \frac{3\sqrt{2g}QL}{4C^2 RBy^{3/2}} \quad (17)$$

where C is the Chézy coefficient.

2.2.3. Proposed whole categorization

The proposed categorization is summarized in Equations 17 and 16, using the same category borders presented in Figure 1 and summarized in Table 1.

3. RESULTS

The proposed parameters are calculated for eight case studies. Each case study defines an open water channel (inspired by the Dutch water system) with different properties (Table 3). In each case the average (working) discharge and water depth is given, and also a possible range in which they can vary. The χ and η values are obtained for the given water depth – discharge range. These ranges and the resulting categories are summarized in Table 4. For calculation of the parameters, the water depth in the middle of the canal is used.

Case	B (m)	S(-)	z (-)	L (m)	n(s/m$^{1/3}$)
1	0.8	0	3	205	0.02
2	6.3	0.00001	5.6	23000	0.02
3	10	0.0002	0.5	40000	0.02
4	2	0	2.3	8225	0.03
5	2	0	2.3	761	0.03
6	8.71	0.00005	3.8	4225	0.02
7	8.24	0.0002	1.7	3334	0.02
8	5.4	0.00004	2.6	13130	0.02

Table 3: Summary of the test cases and channel characteristics, supposing a trapezoidal cross section B is the bottom width, S is the bottom slope, z is the side slope, L is the length of the channel

Case	H_{max} (m)	H_{min}(m)	H_{av} (m)	Q_{min} (m)	Q_{max} (m)	Q_{av} (m)	Type
1	1.66	1.20	1.38	0	10.00	2.00	1
2	3.26	2.18	2.72	0	10.00	2.00	1,3
3	5.48	3.66	4.57	0	10.00	2.00	1,3,4,5
4	1.40	0.93	1.17	0	2.00	1.60	1,3
5	1.40	0.93	1.17	0	2.00	1.60	1
6	4.16	3.44	3.56	0	10.00	2.00	1
7	2.08	1.52	1.89	0	2.60	0.60	1
8	3.62	2.92	3.17	0	10.00	2.00	1

Table 4: Summary of the test cases and channel characteristics, indicating the minimum, maximum and average discharge (Q) and water depth (h)

The results are plotted in Figures 4-11. In order to see the physical behaviour of these channels, the step response is shown in the lower right corner of the figures. These step responses show the downstream depth when upstream a rectangular discharge wave occurs. The magnitude of the discharge wave is the value of q_{max} in Table 4. In all the x-axis the downstream water depth is shown.

Most test cases belong to the short canals. Only case 3 is an exception, and only for high discharges (Figure 6). It can be seen that its response has a different shape - this is categorized as second order response. This canal is the longest, it was expected that it belongs to this group. It can be seen from the division of the groups (Figure 1): if a canal belongs to the longest category it can not have wave reflection. This channel, however, with smaller discharges and higher water levels belongs to group one. It can be seen that the same channel, with high water level and small discharge can have reflecting waves. All other canals are short regarding the categorization.

Reflection waves are very common in the test cases. Most of the cases, except for 2, 3 and 4 have reflecting waves according to the test. In these cases the reflecting waves disappear with the increase of the discharge, however, in all other cases they are present. Looking at the step responses, at most of them the wave reflections are clearly visible as wiggles. There are almost no reflections in Figure 7 - that is what we expected as for higher discharges according to the categorization there are no reflecting waves present.

The wiggles are tiny for cases 1 and 5 and are big for case 2. However, case 1 and 5 are clearly group 1 with resonant waves, and for case 2 it is only true for small discharges and big water levels. This might be unexpected according to the categorization. This issue raises the question if the amplitude of the reflections can be related to the grouping parameter. Also for case 2, the amplitude is 10 cm, and the η number is relatively big, so "not so many reflecting waves" according to the grouping. However, case 6 shows smaller amplitude of reflections (about 3 cm) and according to the parameter it should have "more reflecting waves". There are several reasons for this behavior.

1. There are also different discharge steps for different cases. The channels are also of different sizes.
2. Here the wave damping is calculated, not the resonance peak. Using the resonance peak for the calculations, might balance for the channel width and the discharge step.
3. Some of these channels are highly nonlinear and the current approximation is linear.

Summarising, it can be said about all the categorization based on wave reflections: all responses for the channels belonging to category 1 show the presence of reflecting waves. All the canals that are not in this group, show no reflection on the response.

There are possible improvements for the categorization. Firstly, in these examples the scales were chosen based on the real characteristics of the canals. It would be interesting to increase the extent of the discharge and water depth. Secondly, to categorize the canals based on resonance the damping is chosen. Chosing the resonance peak height would also give a possible characterization. However, in this case, there should be a decision on peak height dividing the two zones. And finally, it is possible to include the nonlinearity and also extend the study to non-trapezoidal canals. The resonance wave categorization is based on theoretical damping. This value could be obtained by doing numerical auto-tune variation tests using a distributed solution of the Saint-Venant equations.

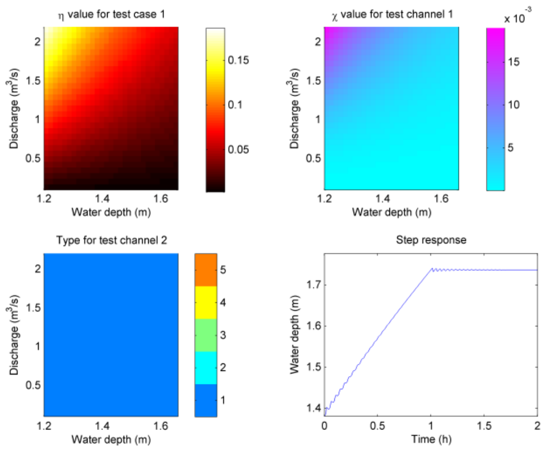

Figure 4: Case 1

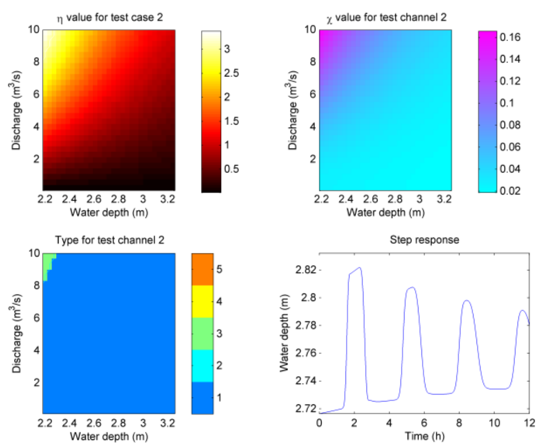

Figure 5: Case 2

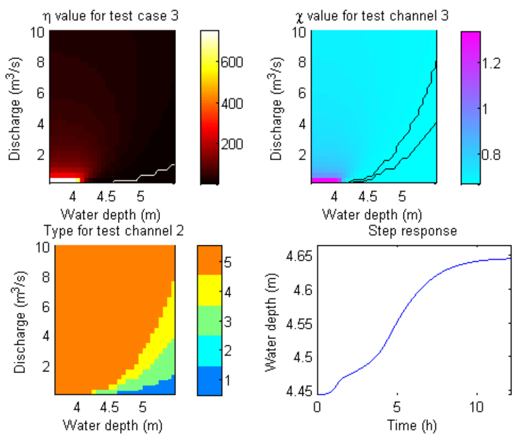

Figure 6: Case 3

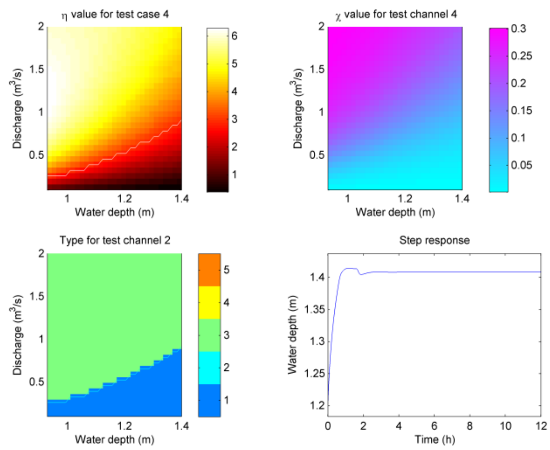

Figure 7: Case 4

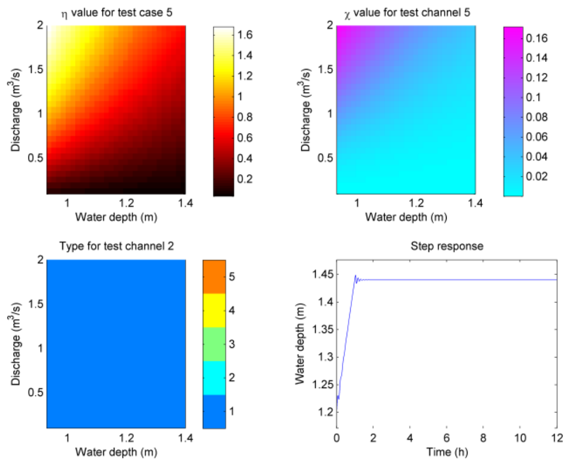

Figure 8: Case 5

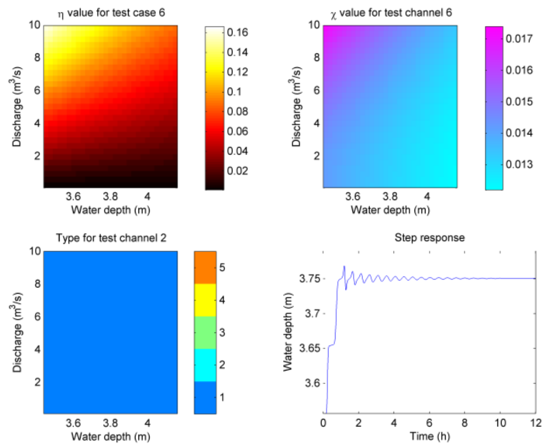

Figure 9: Case 6

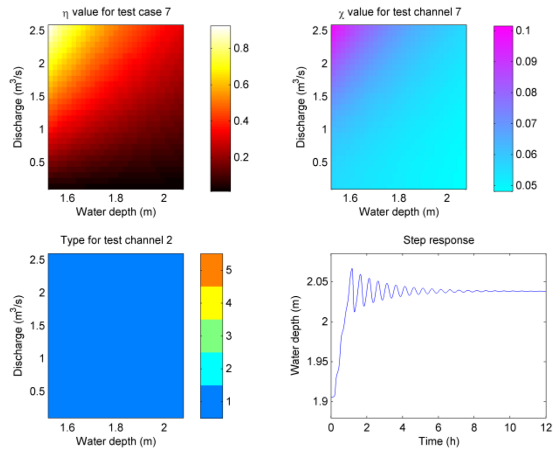

Figure 10: Case 7

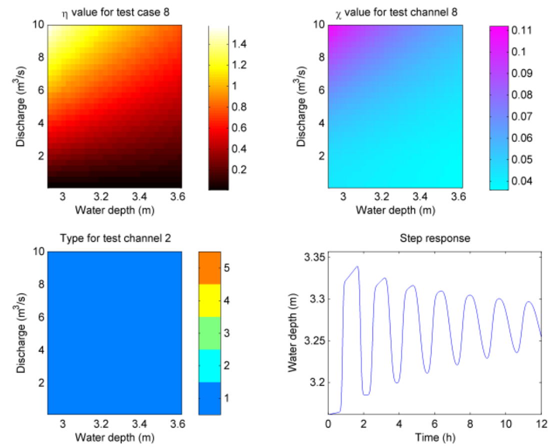

Figure 11: Case 8

4. DISCUSSION

This paper presents a categorization of open channels based on the hydraulic behaviour of the open channel. This categorization can be the base of the choice of simple models. As each category corresponds to a phenomenon, it might be possible to find a one-to-one correspondence between these categories and possible choice of simple models. There are several works addressing the model choice and its limitations [7,6]. In the following section possible suggestions for model choice for each category is given. (Category 2 is not mentioned as it is very rare according to [Baume and Sau, 1997].) At this stage, they are just possible ideas that should be further investigated, and by no means of guidelines.

Category 1, resonant channel: If the travelling waves are significant, the model of this type of channel should contain the water level changes due to wave reflections. For this reason it could be modelled with a linearised diffusive wave model plus inertia term. (With other words the Saint-Venant equations without advection term).

Category 3, non-resonant channel with first order without delay behaviour: This type of channel is short, but not resonant. An Integrator Delay (ID) [Schuurmans et al., 1995] or an Integrator Delay Zero (IDZ) [Litrcio and Fromion, 2004a] type model might be good enough to capture their behaviour.

Category 4, non-resonant channel with first order without delay behaviour: This canal type is longer than type 3, but might still be enough to use and ID or IDZ type of model.

Category 5, non-resonant, second order plus delay: Based on the categorization for length proposed in [Litrico and Georges, 1999], four moments can be matched, a second order model with delay based on the diffusive wave model is a good approximation. Due to the length of the canal the ID and IDZ type models might not give a good match.

These are suggestions, and they just give an idea how the complete categorization and guidance of model choice would look like. The choices should be well-defined, and some test cases should show the relevancy of the answers. Also the choices should be tested on the frequency domain as they will be applied in closed loop.

5. CONCLUSION

A new categorization of open water channels is proposed. This categorization is the combination of existing categorization and its extension to general case. The extension of the categorization is demonstrated on eight test cases, in each case the step responses are shown and the corresponding categories. They demonstrate that each category represents the expected behaviour. With the help of the categorization it is easier to find a corresponding simple hydraulic model for the open water channels. These models can be used for example for automatic control purposes.

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6. APPENDIX A: WAVE PROPAGATION FORMULAS

The condition for the presence of the resonance waves given by [van Overloop et al., 2014] is very similar to the condition given by Baume et al. [1998] and the second formula can be derived as a special case from the first one. It is possible to show that (4) is a special case of (13). Repeating the formula for damping proposed by [van Overloop et al., 2014]:

$$\zeta = \frac{\sqrt{2g}QL}{4C^2RBy^{3/2}}. \tag{17b}$$

Using the Manning equation for uniform flow, the Chézy coefficient can be substituted:

$$\zeta_u = \frac{\sqrt{2g}QLRS_b}{4v^2RBy^{3/2}}. \tag{17c}$$

Assuming rectangular cross section the following formula can be written:

$$\zeta_{ru} = \frac{1}{\sqrt{8}}\frac{cLS_b}{vy}. \tag{17d}$$

The final formula is very similar to the formula given by (4) for η. The difference is a *(1-v/c)* factor and $1/\sqrt{8}$. The factor *(1-v/c)* is close to 1 in case of high wave speeds. The factor $1/\sqrt{8}$ is close to 1/3, which is the difference in the given threshold.

7. APPENDIX B: C AND D FOR NON-UNIFORM FLOW, FOR TRAPEZOIDAL CROSS SECTION

The parameters C_p and D_p are obtained from the linearization of the diffusive wave equations. The following equations, (mentioned in the text as Equation 14 and 15) are repeated here for simplicity:

$$C_p = \frac{1}{T^2 \partial S_f / \partial Q}\left[\frac{\partial T}{\partial x} - \frac{\partial (TS_f)}{\partial z}\right] \tag{18}$$

$$D_p = \frac{1}{T(\partial S_f / \partial Q)}. \tag{19}$$

where Q [m3/s] is the discharge, S_f [-] is the friction slope, z is the elevation and x is the axis parallel to the channel and T [m] is the top width can be calculated as:

$$T = B + 2zy \tag{20}$$

where B [m] is the bottom width, y [m] is the water depth and z [-] is the side slope. The expression for the friction slope S_f [-] is obtained from the Manning equation:

$$S_f = \frac{Q^2 n^2}{A^2 R^{4/3}} \tag{21}$$

where Q [m3/s] is the discharge, n is the Manning's coefficient [s/m1/3], A [m 2] is the cross sectional area and can be obtained as

$$A = By + zy^2. \tag{22}$$

where y [m] is the water depth and z [-] is the side slope for a trapezoidal cross section shown in Figure 12. R [m] is the hydraulic radius and obtained as

$$R = \frac{A}{P} \tag{23}$$

where P [m] is the wet perimeter. It can be calculated for a trapezoidal cross section as:

$$P = B + 2y\sqrt{1+z^2}. \tag{24}$$

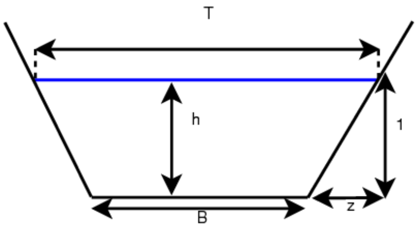

Figure 12: Trapezoidal cross section with notations

7.1.1. Calculation of C

The partial derivative of the friction slope is necessary to calculate C:

$$\frac{\partial S_f}{\partial Q} = \frac{\partial \left[\dfrac{Q^2 n^2}{A^2 R^{4/3}} \right]}{\partial Q} = \frac{2S_f}{Q} \tag{25}$$

The second partial derivative needed for C is the derivative of the top width with respect to x:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y}\frac{\partial y}{\partial x} \tag{26}$$

Finally, we need the following partial to calculate C:

$$\frac{\partial (TS_f)}{\partial z} = \frac{\partial (TS_f)}{\partial y}\frac{\partial y}{\partial z} = \left(T \underbrace{\frac{\partial S_f}{\partial y}}_{(P_1)} + S_f \underbrace{\frac{\partial T}{\partial y}}_{(P_2)} \right) \underbrace{\frac{\partial y}{\partial z}}_{(P_3)} = (TP_1 + S_f P_2)P_3. \tag{27}$$

The first partial (1) appearing in (27) is:

$$
\begin{aligned}
P_1 = \frac{\partial S_f}{\partial y} \quad &= \frac{\partial (Q^2 n^2 A^{-2} R^{-4/3})}{\partial y} = \\
&= Q^2 n^2 \, \frac{\partial (A^{-2} R^{-4/3})}{\partial y} = \\
&= Q^2 n^2 \left(R^{-4/3} \, \frac{\partial (A^{-2})}{\partial y} + A^{-2} \, \frac{\partial (R^{-4/3})}{\partial y} \right) \\
&= Q^2 n^2 \left(R^{-4/3} (-2) A^{-3} (B + 2zy) + A^{-2} (-4/3) R^{-7/3} \, \frac{\partial R}{\partial y} \right) \\
&= Q^2 n^2 \left(R^{-4/3} (-2) A^{-3} (B + 2zy) + A^{-2} (-4/3) R^{-7/3} \, \frac{(B + 2zy)P - 2A\sqrt{1+z^2}}{P^2} \right) \\
&= Q^2 n^2 \left(-2 \frac{(B + 2zy)}{R^{4/3} A^3} - 4/3 \frac{(B + 2zy)P - 2A\sqrt{1+z^2}}{P^2 A^2 R^{7/3}} \right) \\
&= Q^2 n^2 A^{-2} R^{-4/3} \left(-2 \frac{(B + 2zy)}{A} - 4/3 \frac{(B + 2zy)P - 2A\sqrt{1+z^2}}{P^2 R} \right) \\
&= S_f \left(-2 \frac{(B + 2zy)}{A} - 4/3 \frac{(B + 2zy)P - 2A\sqrt{1+z^2}}{PA} \right) \\
&= S_f \left(\frac{-2(B + 2zy) - 4/3(B + 2zy) + 8/3A/P\sqrt{1+z^2}}{A} \right) \qquad (28) \\
&= S_f \left(\frac{-10/3(B + 2zy) + 8/3R\sqrt{1+z^2}}{A} \right)
\end{aligned}
$$

The second partial (2) appearing in (27) is:

$$P_2 = \frac{\partial T}{\partial y} = 2z. \qquad (29)$$

The calculation of the third partial (3) appearing in (27) is equal to 1 measuring the elevation from the lowest water depth point and taking into account only the points along the water depth:

$$P_3 = \frac{\partial y}{\partial z} = 1. \qquad (30)$$

This is a valid assumption as we are only interested in this ration where the water depth is bigger than zero. Finally, C_p can be calculated by substituting (25, 26, 27) into (6).

$$
\begin{aligned}
C_p &= \frac{1}{T^2 \partial S_f / \partial Q}\left[\frac{\partial T}{\partial x} - \frac{\partial (T S_f)}{\partial z}\right] \\
&= \frac{Q}{2T^2 S_f}\left[2z\frac{S_b - S_f}{1-Fr^2} - (TP_1 + S_f P_2)P_3\right] = \\
&= \frac{Q}{2T^2 S_f}\left[2z\frac{S_b - S_f}{1-Fr^2} - 2zS_f - TP_1\right] = \\
&= \frac{Q}{2T^2 S_f}\left[2z\frac{S_b - S_f}{1-Fr^2} - 2zS_f - TSf\left(\frac{-10/3(B+2zy) + 8/3R\sqrt{1+z^2}}{A}\right)\right]
\end{aligned} \tag{31}
$$

7.1.2. Calculation of D

Substitution 25 into 7

$$
D(Q,z,x) = \frac{1}{T(\partial S_f / \partial Q)} = \frac{Q}{2TS_f} \tag{32}
$$