Bayesian Inference of Spatially-Varying Manning's *n* Coefficients in the Coastal Ocean Using a Generalized Karhunen-Loève Expansion and Polynomial Chaos

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Introduction

- The shallow water model is composed of the depth-integrated Navier-Stokes equations
  - *Continuity Equation:* $$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(Q_x) + \frac{\partial}{\partial y}(Q_y) = 0$$
  - *Momentum Equation:*

$$\frac{\partial Q_x}{\partial t} + \frac{\partial U Q_x}{\partial x} + \frac{\partial V Q_x}{\partial y} - f Q_y = -gH \frac{\partial[\zeta + P_s/g\rho_0 - \alpha\eta]}{\partial x} + \frac{\tau_{sx}}{\rho_0} - \frac{\tau_{bx}}{\rho_0} + M_x - D_x - B_x$$

$$\frac{\partial Q_y}{\partial t} + \frac{\partial U Q_y}{\partial x} + \frac{\partial V Q_y}{\partial y} - f Q_x = -gH \frac{\partial[\zeta + P_s/g\rho_0 - \alpha\eta]}{\partial y} + \frac{\tau_{sy}}{\rho_0} - \frac{\tau_{by}}{\rho_0} + M_y - D_y - B_y.$$

Bottom stress terms

Introduction

The bottom stress components in the momentum equation are defined through the coefficient

$$K_{slip} = c_f |\mathbf{u}|$$

Then c_f is determined using Manning's n formulation

$$c_f = \frac{gn^2}{H^{1/3}}$$

Manning's n Coefficient

- Empirically derived
- Depends on surface characteristics
- Spatially variable

1. **Bayesian Inference**
2. **Coordinate transformation for Uncertain Correlation Function**
3. **PC surrogate model**
4. **Manning's *n* field inference**

Inference of parameter field

We want to infer a parameter field $M \in L_2(\Omega)$, from

- a set of observations $\boldsymbol{d} \in \mathbb{R}^m$ of a given process,
- a model $\boldsymbol{u}(M) \in \mathbb{R}^m$ that predicts the observation,
- the Bayesian rule to update our knowledge of M.

$$p(M, \sigma_o^2 | \boldsymbol{d}) \propto p(\boldsymbol{d} | M, \sigma_o^2) p_M(M) p_o(\sigma_o^2)$$

- $p(\boldsymbol{d} | M, \sigma_o^2)$ is the likelihood of the observations,
- $p_M(M)$ is the Gaussian field's prior,
- σ_o^2 is an error model hyper-parameter with prior of $p_o(\sigma_o^2)$.

Classical choices are i.i.d. model errors with Gaussian distribution $N(0, \sigma_o^2)$ leading to

$$p(\boldsymbol{d} | M, \sigma_o^2) = \prod_{i=1}^{m} p_\epsilon(d_i - u_i(M), \sigma_o^2), \quad p_\epsilon(x, \sigma_o^2) \doteq \frac{1}{\sqrt{2\pi\sigma_o^2}} \exp\left[-\frac{x^2}{2\sigma_o^2} \right]$$

with uninformative Jeffrey's prior for σ_o.

Gaussian field's prior

We shall consider prior M that are centered Gaussian processes with covariance function $\mathcal{C}(\boldsymbol{x}, \boldsymbol{x}')$.

The prior $M(\boldsymbol{x})$ can then be decomposed in Principal Orthogonal Components (KL decomposition),

$$C(\boldsymbol{x}, \boldsymbol{x}') = \sum_{k=1}^{\infty} \lambda_k \phi_k(\boldsymbol{x}) \phi_k(\boldsymbol{x}'), \quad M(\boldsymbol{x}) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \Phi_k(\boldsymbol{x}) \eta_k,$$

where the η_k's are iid standard Gaussian random variables.

Upon truncation of the expansion of M to its K dominant terms,

$$M(\boldsymbol{x}) \approx M_K(\boldsymbol{x}) = \sum_{k=1}^{K} \sqrt{\lambda_k} \Phi_k(\boldsymbol{x}) \eta_k,$$

Inference problem can for the stochastic coordinates η_k's :

$$p(\boldsymbol{\eta}, \sigma_o^2 | \boldsymbol{d}) \propto p(\boldsymbol{d} | \boldsymbol{\eta}, \sigma_o^2) p_\eta(\boldsymbol{\eta}) p_o(\sigma_o^2),$$

with

$$p_\eta(\boldsymbol{\eta}) = \frac{1}{(2\pi)^{K/2}} \exp - \|\boldsymbol{\eta}\|^2 / 2, \quad p(\boldsymbol{d} | \boldsymbol{\eta}, \sigma_o^2) = \prod_{i=1}^{m} p_\epsilon(d_i - u_i(\boldsymbol{\eta}), \sigma_o^2).$$

Uncertainty in the covariance function

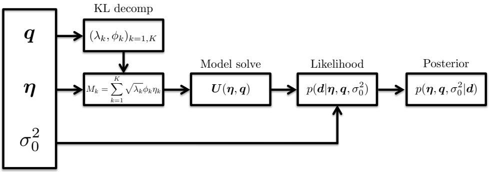
The selection of the covariance function affects the inference procedure and C is in general uncertain.
$\Rightarrow$ families of covariance functions $\mathcal{C}(\boldsymbol{q})$ with hyper-parameters $\boldsymbol{q}$, with prior $p_q(\boldsymbol{q})$ (also inferred).

Following this approach, we write

$$M(\boldsymbol{x},\boldsymbol{q}) \approx M_K(\boldsymbol{x},\boldsymbol{q}) = \sum_{k=1}^{K} \sqrt{\lambda_k(\boldsymbol{q})}\Phi_k(\boldsymbol{x},\boldsymbol{q})\eta_k,$$

where the η_k's are still i.i.d. standard Gaussian random variables and $(\lambda_k(\boldsymbol{q}), \Phi_k(\boldsymbol{q}))$ are the dominant proper elements of $\mathcal{C}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{q})$.

$$p(\boldsymbol{\eta},\boldsymbol{q},\sigma_o^2|\boldsymbol{d}) \propto p(\boldsymbol{d}|\boldsymbol{\eta},\boldsymbol{q},\sigma_o^2)p_\eta(\boldsymbol{\eta})p_q(\boldsymbol{q})p_o(\sigma_o^2).$$



- many KL decomposition
- many model solves

- change of coordinate
- Use of PC surrogate

Reference Basis

For any covariance parameters $\boldsymbol{q}$, the elements of the KL expansion are solution of

$$\int_\Omega \mathcal{C}(\boldsymbol{x}, \boldsymbol{x}')\Phi_k(\boldsymbol{x}', \boldsymbol{q})d\boldsymbol{x}' = \lambda_k(\boldsymbol{q})\Phi_k(\boldsymbol{x}, \boldsymbol{q}), \quad (\Phi_k, \Phi_k)_X = 1.$$

We observe that $\{\Phi_k(\boldsymbol{q})\}$ is a CONS of $L_2(\Omega)$.

It suggests the introduction of a reference orthonormal basis $\{\bar{\Phi}_k\}$, defined for a prescribed reference covariance function $\overline{\mathcal{C}}$, and to project $M_k(\boldsymbol{q})$ onto this reference subspace.

For a finite dimensional reference basis (with K modes for simplicity), it comes

$$M_k(\boldsymbol{q}) = \sum_{k=1}^{K} \tilde{\Phi}_k(\boldsymbol{q})\eta_k \approx \overline{M}_K = \sum_{k=1}^{K} \bar{\Phi}_k \bar{\eta}_k(\boldsymbol{q}), \quad \bar{\boldsymbol{\eta}}(\boldsymbol{q}) = \mathcal{B}(\boldsymbol{q})\boldsymbol{\eta}.$$

Regarding the selection of the reference basis :

- select of particular hyper-parameter value : $\overline{\mathcal{C}} = \mathcal{C}(\bar{\boldsymbol{q}})$
- use the $\boldsymbol{q}$-averaged covariance function,

$$\bar{\mathcal{C}} = \langle \mathcal{C} \rangle = \int \mathcal{C}(\boldsymbol{q})p_q(\boldsymbol{q})d\boldsymbol{q}.$$

The latter choice is optimal in terms of representation error (averaged over $\boldsymbol{q}$).

PC surrogate : motivation

Sampling of the posterior $p(\boldsymbol{\eta}, \boldsymbol{q}, \sigma_o^2 | \boldsymbol{d})$ involves many resolution of the forward model to predict the observation $\boldsymbol{u}(\boldsymbol{\eta}, \boldsymbol{q})$.

To accelerate this step, the use of polynomial surrogates (PC expansions) was proposed by Marzouk, Najm, *et al* :

$$\boldsymbol{u}(\boldsymbol{\eta}, \boldsymbol{q}) \approx \sum_{\alpha=0}^{P} \boldsymbol{u}_\alpha \Psi_\alpha(\boldsymbol{\eta}, \boldsymbol{q}),$$

where the Ψ_α's are orthogonal polynomials and the PC expansion is truncated at some order r.

The PC expansion is computed in an off-line stage.

We propose an alternative approach, relying on coordinate transformation :

$$\boldsymbol{u}(\boldsymbol{\eta}, \boldsymbol{q}) \approx \hat{u}(\boldsymbol{\xi}(\boldsymbol{\eta}, \boldsymbol{q})) = \sum_{\alpha=0}^{P} \boldsymbol{u}_\alpha \Psi(\boldsymbol{\xi}(\boldsymbol{\eta}, \boldsymbol{q})),$$

where the random vector $\boldsymbol{\xi}$ has the same dimension as $\boldsymbol{\eta}$, that is K.

PC surrogate

It can be shown that we can approximate $\bar{\boldsymbol{\eta}} \mapsto \boldsymbol{u}(\bar{\boldsymbol{\eta}})$ using the reference Gaussian field

$$\overline{M}_K^{\mathrm{PC}}(\boldsymbol{\xi}) = \sum_{k=1}^{K} \sqrt{\overline{\lambda}_k}\,\overline{\phi}_k \xi_k, \quad \boldsymbol{\xi} \mapsto \hat{\boldsymbol{u}}(\boldsymbol{\xi}) \approx \sum_{\alpha=0}^{P} \hat{\boldsymbol{u}}_\alpha \Psi_\alpha(\boldsymbol{\xi}),$$

where the ξ_k's are independent standard Gaussian random variables. Then

$$\boldsymbol{u}(\boldsymbol{\eta}, \boldsymbol{q}) \approx \sum_{\alpha=0}^{P} \hat{\boldsymbol{u}}_\alpha \Psi_\alpha(\boldsymbol{\xi}(\boldsymbol{\eta}, \boldsymbol{q})), \quad \boldsymbol{\xi}(\boldsymbol{\eta}, \boldsymbol{q}) = \tilde{\mathcal{B}}(\boldsymbol{q})\boldsymbol{\eta}, \quad \tilde{\mathcal{B}}_{kl}(\boldsymbol{q}) = \begin{cases} \dfrac{\mathcal{B}_{kl}(\boldsymbol{q})}{\sqrt{\overline{\lambda}_k}}, & \overline{\lambda}_k/\overline{\lambda}_1 > \kappa, \\ 0, & \text{otherwise.} \end{cases}$$

Sampling flow-chart

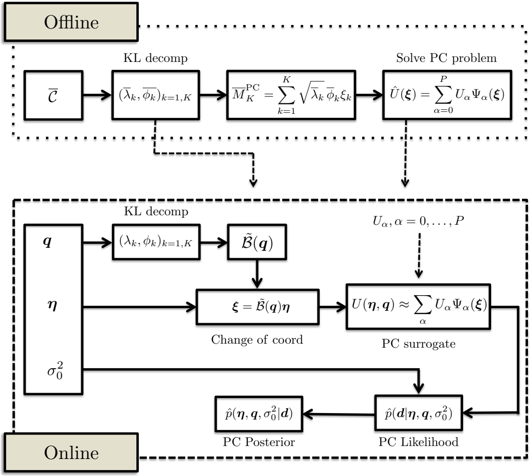


FIGURE: Offline step (surrogate construction) of the accelerated MCMC sampler and Online step of the PC surrogate based evaluation of the posterior.

ADCIRC

Inference for "true" Manning's *n* field :

- ADvanced CIRCulation (ADCIRC) solves the shallow water equations on an unstructured grid, discretized by a first-order continuous Galerkin finite element.
- The time derivatives computed with centered finite differences in GWCE and forward differences in the momentum equations.
- ADCIRC was intensively validated, e.g. Hurricanes Betsy (1965), Ivan (2004), Dennis (2004), Katrina (2005), Rita (2005), Gustav (2008) and Ike (2008)

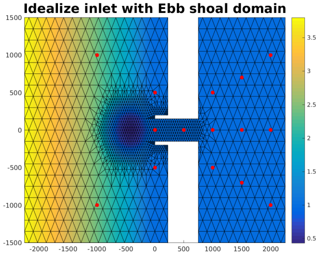


FIGURE: Idealized inlet with ebb shoal domain.

OSSEs

Observation simulation system experiments (OSSEs) :

- Synthetic water elevation data are extracted from an ADCIRC run.
- Manning's *n* field used in reference run is considered truth.
- We attempt to recover Manning's based on the data and ADCIRC, using a generalized KL expansion and PC-MCMC.

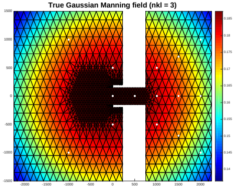
Observations are measurements of $U(x, t)$ (water elevation) at several locations in space and time, perturbed with i.i.d. $\epsilon_i \sim N(0, \sigma_\epsilon^2 = 0.01)$.

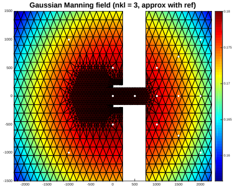
For prior, we use $M \sim \mathcal{GP}(0, \mathcal{C}(\boldsymbol{q}))$, with Gaussian covariance $\mathcal{C}(\boldsymbol{q})$ and hyper-parameter $\boldsymbol{q} = \{l\}$:

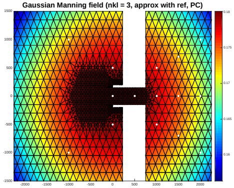
- $l \sim U[1000, 4000]$,

Offline : reconstruction of Manning's *n* field

We set $K = 3$, true normalized $l = 0.085$ and true coordinates $\{\eta_1, \eta_2, \eta_3\} = \{1.73, 0.26, 0.04\}$.

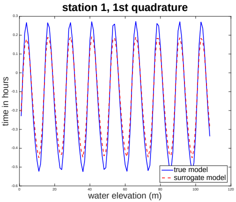


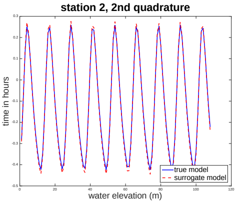


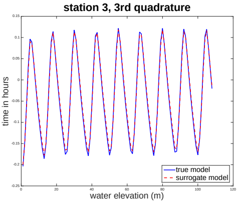


Offline : PC surrogate of the ADCIRC model

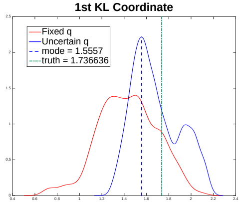
Based on reference q, number of stochastic dimension equal to 3 and $r = 6$.

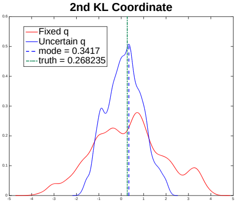


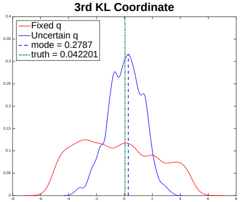


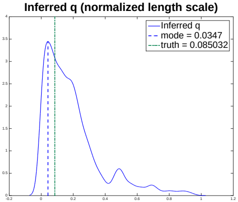


Online : MCMC inference results

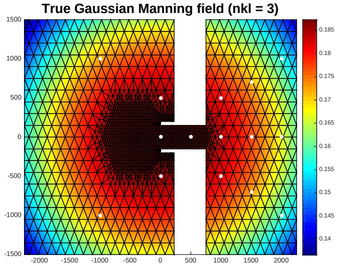


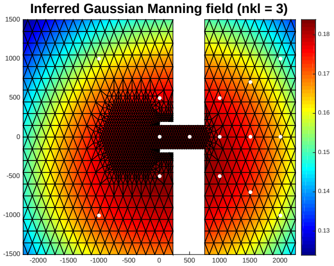






Inference : True Manning's field vs. inferred field





Conclusion & Future work

- Effective treatment of covariance hyper-parameters
- Generic PC construction for the surrogate
- Accelerate both coordinate transformation and likelihood sampling using PC surrogate
- Successfully application of generalized KL and PC for parameter inference to large-scale coastal ocean

Further possibilities

- Treats the prior in the Baysian inference directly instead of resorting to coordinate transformation approach (which can be expensive for large-scale system)