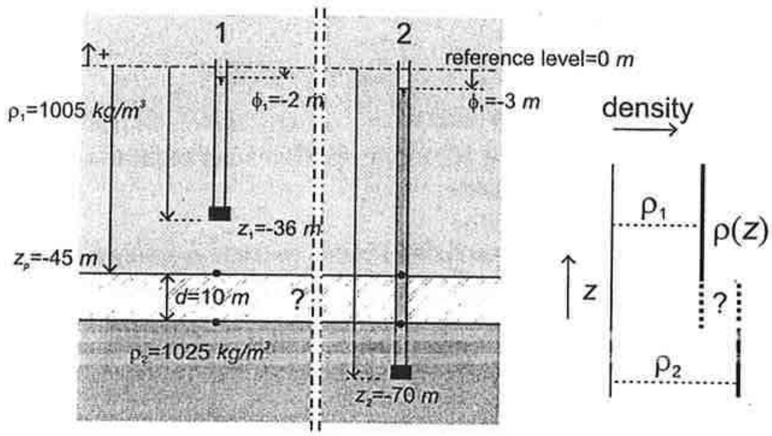
Exercise 3.1: Vertical flow through the aquitard ?

Does vertical flow occur through the resistance layer as schematised in the situation given in the figure ? Note that both observation wells are located at one place.



Solution exercise 3.1: vertical flow through the aquitard ?

Reference level at $z = -45m$

Superposition: resistance layer is filled with brackish groundwater (ρ=1005 kg/m^3):

$$\phi_{f,1}^{z=-45} = -45 + \frac{1005}{1000}34 + \frac{1}{1000g}\int_{-45}^{-36} 1005gdz = -45 + 34 \cdot 1.005 + 1.005 \cdot [-36 + 45]$$

$$\phi_{f,1}^{z=-45} = -45 + 34.170 + 9.045 = -1.785m$$

$$\phi_{f,2}^{z=-45} = -45 + \frac{1025}{1000}67 - \frac{1}{1000g}\int_{-55}^{-45} 1005gdz - \frac{1}{1000g}\int_{-70}^{-55} 1025gdz$$

$$\phi_{f,2}^{z=-45} = -45{+}68.675{-}1.005(-45{+}55){-}1.025(-55{+}70) = -45{+}68.675{-}10.05{-}15.375 = -1.750m$$

$$\Delta\phi = 0.035m$$

Appearently, there is a small upward flow of groundwater: this means that the superposition is wrong. Now suppose that the resistance layer is filled with saline groundwater (ρ=1025 kg/m^3):

$$\phi_{f,1}^{z=-45} = -45 + \frac{1005}{1000}34 + \frac{1}{1000g}\int_{-45}^{-36} 1005gdz = -1.785m$$

$$\phi_{f,2}^{z=-45} = -45 + \frac{1025}{1000}67 - \frac{1}{1000g}\int_{-70}^{-45} 1025gdz = -45 + 67 \cdot 1.025 - 1.025(-45+70)$$

$$\phi_{f,2}^{z=-45} = -45 + 68.675 - 25.625 = -1.950m$$

$$\Delta\phi = -0.165m$$

Now, a downward flow of groundwater occurs, which implies that this superposition is wrong either. This means that the sharp interface between brackish and saline groundwater is located in the resistance layer.

Reference level at $z = -55m$

Superposition: resistance layer is filled with brackish groundwater (ρ=1005 kg/m^3):

$$\phi_{f,1}^{z=-55} = -55 + \frac{1005}{1000}34 + \frac{1}{1000g}\int_{-55}^{-36} 1005gdz = -55 + 34 \cdot 1.005 + 1.005 \cdot [-36+55]$$

$$\phi_{f,1}^{z=-55} = -55 + 34.17 + 19.095 = -1.735m$$

$$\phi_{f,2}^{z=-55} = -55 + \frac{1025}{1000}67 - \frac{1}{1000g}\int_{-70}^{-55} 1025gdz = -55 + 68.675 - 1.025(-55+70)$$

$$\phi_{f,2}^{z=-55} = -55 + 68.675 - 15.375 = -1.700m$$

$$\Delta = 0.035m$$

There is an upward flow of groundwater: this means that the superposition is wrong. Now suppose that the resistance layer is filled with saline groundwater (ρ=1025 kg/m^3):

$$\phi_{f,1}^{z=-55} = -55 + \frac{1005}{1000}34 + \frac{1}{1000g}\int_{-45}^{-36} 1005gdz + \frac{1}{1000g}\int_{-55}^{-45} 1025gdz$$

$$\phi_{f,1}^{z=-55} = -55 + 34.17 + 9.045 + 10.25 = -1.535m$$

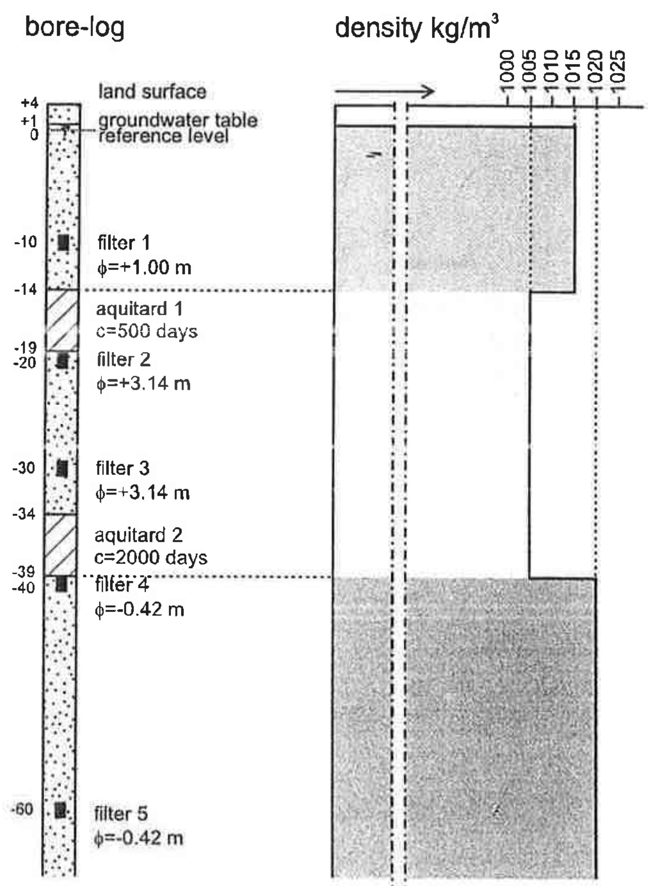
$$\phi_{f,2}^{z=-55} = -55 + \frac{1025}{1000}67 - \frac{1}{1000g}\int_{-70}^{-55} 1025gdz = -1.700m$$

$$\Delta = -0.165m$$

Here, groundwater flows downward, which implies that this superposition is wrong either. This means that the sharp interface between brackish and saline groundwater is located in the resistance layer. Note that the difference in freshwater head is equal for the reference levels on different places ($z = -45$ m and $z = -55$ m), whereas the absolute values of the freshwater heads differ.

Exercise 3.2: Determination of vertical groundwater flow through the aquitards.

Calculate the vertical groundwater flow through the two aquitards (e.g. clayey layers) for the schematisation as given in the figure.



Solution exercise 3.2: determination of vertical groundwater flow through the aquitards.

Vertical flow through aquitard 1
Level to determine freshwater head: $z_p = -19\ m$
Use filter 1 at $z = -10\ m$

$$\phi_{f,1}^{z_p=-19} = -19 + \frac{1015}{1000}11 + \frac{1}{1000}\int_{-14}^{-10} 1015 dz + \frac{1}{1000}\int_{-19}^{-14} 1005 dz$$

$$\phi_{f,1}^{z_p=-19} = -19 + 1.015 \cdot 11 + 1.015 \cdot [-10+14] + 1.005 \cdot [-14+19] = 1.25\,m$$

Use filter 2 at $z = -20\ m$

$$\phi_{f,2}^{z_p=-19} = -19 + \frac{1005}{1000} 23.14 - \frac{1}{1000} \int_{-20}^{-19} 1005 dz$$

$$\phi_{f,2}^{z_p=-19} = -19 + 1.005 \cdot 23.14 - 1.005(-19+20) = 3.25\,m$$

$$\Delta\phi^{aquitard1} = 2.00\,m$$

$$q^{aquitard1} = \frac{2000mm}{500day} = 4.0\ mm/day \text{ upward}$$

Vertical flow through aquitard 2
Level to determine freshwater head: $z_p = -39\ m$
Use filter 3 at $z = -30\ m$

$$\phi_{f,3}^{z_p=-39} = -39 + \frac{1005}{1000} 33.14 + \frac{1}{1000} \int_{-39}^{-30} 1005 dz$$

$$\phi_{f,3}^{z_p=-39} = -39 + 1.005 \cdot 33.14 + 1.005 \cdot [-30+39] = 3.35\,m$$

Use filter 4 at $z = -40\ m$

$$\phi_{f,4}^{z_p=-39} = -39 + \frac{1020}{1000} 39.58 - \frac{1}{1000} \int_{-40}^{-39} 1020 dz$$

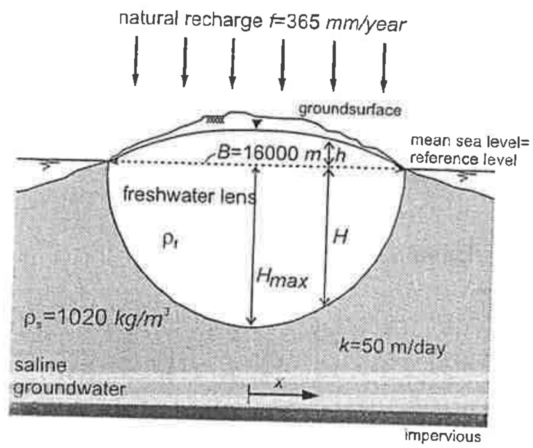
$$\phi_{f,4}^{z_p=-39} = -39 + 1.020 \cdot 39.58 - 1.020(-39+40) = 0.35\,m$$

$$\Delta\phi^{aquitard2} = 3.00\,m$$

$$q^{aquitard1} = \frac{3000mm}{2000day} = 1.5\ mm/day \text{ downward}$$

Exercise 4.1: Freshwater lens in an one dimensional situation.

Consider an infinite long strip of sand-dune with a cross-section as given in the figure. The sea level is constant.



4.1.1 Derive the global analytical equations for h, H and q (thus including the constants $C1$ and $C2$) for the interface between fresh and saline in a phreatic aquifer (one-dimensional situation), as shown in the figure.

4.1.2 Determine the shape of the freshwater lens (viz. H and h as a function of x), based on the answer of question 4.1.1 and the parameters in the figure.

4.1.3 Determine h and H in the middle of the dune and q at the boundary of the freshwater lens.

4.1.4 Explain what will happen with the freshwater lens if no recharge occurs during a long period.

4.1.5 Now assume an aquiclude (impervious layer) at a depth D of 146 m below mean sea level. Where ('at x=...') will the freshwaterlens reach this aquiclude ?

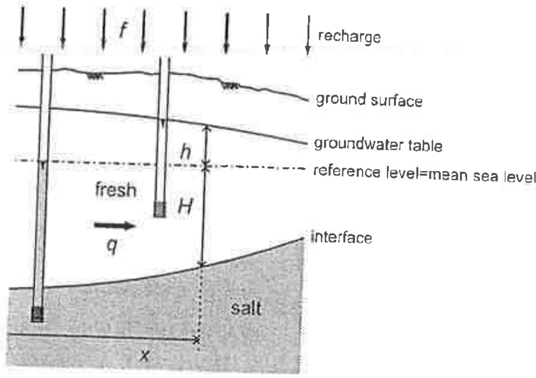
Solution exercise 4.1: freshwater lens in an one dimensional situation

4.1.1 The three governing equations are (see the figure as well as the lecture notes):

(I) Darcy: $$q = -k(H+h)\frac{dh}{dx} \tag{0.1}$$

(II) Continuity: $$dq = f dx \tag{0.2}$$

(III) BGH: $$h = \alpha H \tag{0.3}$$



The analytical formulae are as follows:

$$H = \sqrt{\frac{-fx^2 - 2C_1x + 2C_2}{k(1+\alpha)\alpha}} \tag{0.4}$$

$$h = \alpha H \tag{0.5}$$

$$q = fx + C_1 \tag{0.6}$$

4.1.2 The boundary conditions are:

$$x = 0 : q = 0 \Longleftrightarrow C_1 = 0 \tag{0.7}$$

$$x = 0.5B : H = 0 \Longleftrightarrow C_2 = fB^2/8 \tag{0.8}$$

The depth of the fresh-salt interface is:

$$H = \sqrt{\frac{f\,(0.25\,B^2 - x^2)}{k\,(1+\alpha)\,\alpha}} = \sqrt{9.797 \cdot 10^{-4}(8000^2 - x^2)} \tag{0.9}$$

$$h = \alpha H = 0.02H \tag{0.10}$$

$$q = fx = 0.365x \tag{0.11}$$

4.1.3

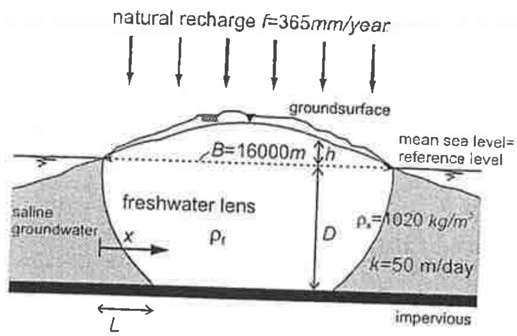
$$\text{at } x = 0 : H = 250.40\,m \tag{0.12}$$

$$\text{at } x = 0 : h = 0.02H = 5.01\,m \tag{0.13}$$

$$\text{at } x = 8000\,m : q = 8.0\,m^2/day \tag{0.14}$$

4.1.4 The freshwater lens will disappear.

4.1.5 Note that the x-axes is moved to the boundary of the freshwater lens to determine the boundary conditions properly. The reason is that between $L < x < B - L$, no sharp interface is present, and thus, the Badon Ghijben-Herzberg principle and



the accompanying analytical formulae are not applicable anymore. The analytical formula for the interface H (at $x < L$) is:

$$H = \sqrt{\frac{-fx^2 - 2C_1x + 2C_2}{k(1+\alpha)\alpha}} \tag{0.15}$$

The boundary conditions are:

$$x = 0 : q = -0.5fB \Longleftrightarrow C_1 = -0.5fB \tag{0.16}$$

$$x = 0 : H = 0 \Longleftrightarrow C_2 = 0 \tag{0.17}$$

The fresh-salt interface becomes:

$$H = \sqrt{\frac{-fx^2 + fBx}{k(1+\alpha)\alpha}} \tag{0.18}$$

$$\text{at } x = L : H = D = 146\,m \tag{0.19}$$

$$H = D = \sqrt{\frac{-fL^2 + fBL}{k(1+\alpha)\alpha}} \tag{0.20}$$

$$D^2 = \frac{-fL^2 + fBL}{k(1+\alpha)\alpha} \tag{0.21}$$

$$fL^2 - fBL + D^2k(1+\alpha)\alpha = 0 \tag{0.22}$$

The 'abc-formula' gives:

$$L_{1,2} = \frac{Bf \pm \sqrt{B^2f^2 - 4fD^2k(1+\alpha)\alpha}}{2f} \tag{0.23}$$

$$L_{1,2} = 0.5B \pm 0.5\sqrt{B^2 - \frac{4D^2k(1+\alpha)\alpha}{f}} \tag{0.24}$$

$$L_{1,2} = 8000 \pm 6499.4 \tag{0.25}$$

$$L_1 = 1500.5\,m; L_2 = 14499.4\,m \tag{0.26}$$

Exercise 4.2

Gualbert Oude Essink

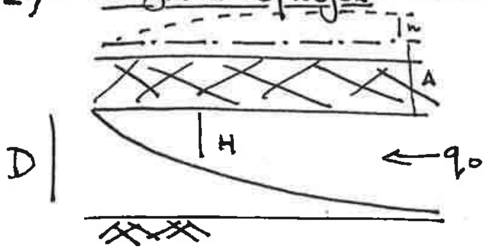
4.2.1) Saltwater wedge up to $x = L$:

$$h_0 = (H + A)\,\alpha \qquad \alpha = \frac{1025 - 1000}{1000} = 0.025$$

$H = D$

$$h_0 = 0.025 \cdot (20 + 30) = \boxed{1.25\text{ m}}$$

4.2.2) Confined aquifer



Darcy: $q = -k\,H\,\frac{dh}{dx}$

Continuity: $q = q_0$

BGH: $h = \alpha(H + A) \Leftrightarrow \frac{dh}{dx} = \alpha\,\frac{dH}{dx}$

$$\Leftrightarrow q_0 = -k\,H\,\alpha\,\frac{dH}{dx} \Leftrightarrow -q_0\,dx = \alpha k\,H\,dH \overset{\frown}{\Leftrightarrow} \tfrac{1}{2}\alpha k H^2 = -q_0 x + C$$

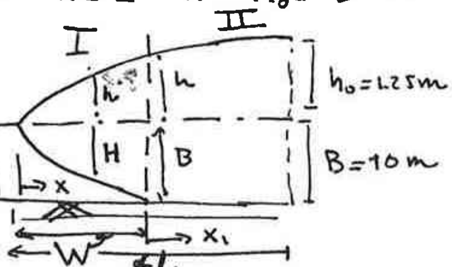
Boundary condition: $x = 0,\ H = 0$: $\quad 0 = 0 + C \Leftrightarrow C = 0$

equation: $\tfrac{1}{2}\alpha k H^2 = -q_0 x$

At $x = L$: $H = H_0 = D \Leftrightarrow \tfrac{1}{2}\alpha k\,D^2 = -q_0 \cdot L \Leftrightarrow \boxed{q_0 = -\frac{\frac{1}{2}\alpha k D^2}{L} = -0.05\ \text{m}^3/\text{m'd}}$

4.2.3)

Part I



Phreatic aquifer

Darcy: $q = -k\,(H + h)\frac{dh}{dx}$

Continuity: $q = q_0$ (no recharge!)

BGH: $h = \alpha H \Leftrightarrow \frac{dh}{dx} = \alpha\,\frac{dH}{dx}$

$$\Leftrightarrow q_0 = -k\,(H + h)\,\alpha\,\frac{dH}{dx} \Leftrightarrow -k\,(1+\alpha)\,H\alpha\,dH = q_0\,dx$$

$$\Leftrightarrow -\tfrac{1}{2}k(1+\alpha)\alpha H^2 = q_0 x + C$$

Boundary condition: $x = 0;\ H = 0$: $\quad 0 = 0 + C \Leftrightarrow C = 0$

equation: $+\tfrac{1}{2}k(1+\alpha)\alpha H^2 = -q_0 x$

At $x = W$: $H = B \Leftrightarrow +\tfrac{1}{2}k(1+\alpha)\alpha B^2 = -q_0 W \Leftrightarrow \boxed{W = \frac{-\frac{1}{2}k(1+\alpha)\alpha B^2}{q_0}}$

Part II No freshwater-salinewater sharp interface.

Darcy: $q = -k\,(B + h)\,\frac{dh}{dx_1}$ $\qquad$ Continuity: $q = q_0$

$$\Leftrightarrow q_0 = -k\,(B + h)\frac{dh}{dx_1} \Leftrightarrow -k\,(B + h)\,dh = q_0\,dx_1$$

$$\Leftrightarrow -kBh - \tfrac{1}{2}kh^2 = q_0\,x_1 + C_1$$

Boundary condition: $x_1 = 0$, $h = \alpha B$

$\Leftrightarrow -k\alpha B^2 - \frac{1}{2}k\alpha^2 B^2 = 0 + C_1$

$\Leftrightarrow -kBh - \frac{1}{2}kh^2 = q_0 x_1 - k\alpha B^2 - \frac{1}{2}k\alpha^2 B^2$

At $x_1 = L - W$: $h = h_0 = 1.25\,m$: now q_0 can be deduced

From part I: $W = -\frac{1}{2}k\alpha(1+\alpha)B^2/q_0$

$\Leftrightarrow -kBh_0 - \frac{1}{2}kh_0^2 = q_0(L - -\frac{1}{2}k\alpha(1+\alpha)B^2/q_0) - k\alpha B^2 - \frac{1}{2}k\alpha^2 B^2$

$\Leftrightarrow -375 - 23.4375 = q_0(3000 + 38.4375/q_0) - 75 - 0.9375$

$k=30$, $B=10$, $h_0=1.25$, $\alpha=0.025$

$\Leftrightarrow 3000\,q_0 = -375 - 23.4375 + 75 + 0.9375 - 38.4375$

$\Leftrightarrow q_0 = \frac{-360.9375}{3000} = -0.1203\ m^3/m\cdot day$

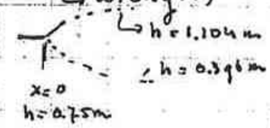
$\Leftrightarrow W = -\frac{1}{2}k\alpha(1+\alpha)B^2/q_0 = \boxed{+319.48\ m}$

4.2.4 Confined aquifer: (question 4.2.2)

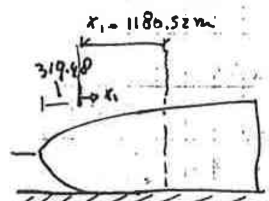
$-\frac{1}{2}\alpha k H^2 = q_0 x \Leftrightarrow H = \pm\sqrt{\frac{-q_0 x}{\frac{1}{2}\alpha k}} \Leftrightarrow h = \pm\sqrt{-\frac{2\alpha q_0}{k}x} + \alpha A$ ($h = \alpha(H+A)$)

At $x = \frac{1}{2}L \Rightarrow x = 1500\,m$; $q_0 = -0.05$ (question 4.2.2)

$h_{x=1500m} = \pm\sqrt{\frac{-2\cdot 0.025 \cdot -0.05}{30}\cdot 1500} + 0.025\cdot 30 = \boxed{1.104\ m}$ (OR $h = 0.396\,m$) ↳ wrong(!)



Phreatic aquifer: (question 4.2.3)



$x = \frac{1}{2}L = 1500\,m$: $X_1 = 1500 - 319.48\,m = 1180.52\,m$.

$\Leftrightarrow h_{x_1=1180.52}$: $-kBh - \frac{1}{2}kh^2 = q_0 \cdot 1180.52 - k\alpha B^2 - \frac{1}{2}k\alpha^2 B^2$

$\Leftrightarrow -15h^2 - 300h - -0.1203 \cdot 1180.52 + k\alpha B^2 + \frac{1}{2}k\alpha^2 B^2 = 0$

$\Leftrightarrow -15h^2 - 300h + 218.5 = 0 \Leftrightarrow 15h^2 + 300h - 218.5 = 0$

$\Leftrightarrow h_{1,2} = \frac{-300 \pm \sqrt{300^2 + 4\cdot 15\cdot 218.5}}{30} \Leftrightarrow \boxed{h_{1,2} = +0.704\ m} / -\sout{20.73}$ nonsense

Difference: $1.104\,m - 0.704\,m = \boxed{0.40\ m}$