Investigating the internal pressure gradient error in σ-coordinate ocean models

Helene Hisken Pedersen
Supervised by Professor Jarle Berntsen

Department of Mathematics, University of Bergen, Norway

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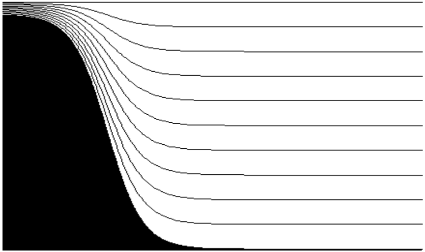
σ-coordinate ocean models

- Vertical coordinates follow the terrain

$$x^* = x \qquad y^* = y \qquad \sigma = \frac{z - \eta}{H + \eta} \qquad t^* = t$$



σ-coordinate ocean models

- Advantageous when dealing with large variations in topography
- Bottom and top boundary layers represented accurately
- E.g. Princeton Ocean Model (POM), Regional Ocean Model System (ROMS), Bergen Ocean Model (BOM)

A problem

The transformation to σ-coordinates introduces an error in the *internal* pressure force

The internal pressure gradient error

- The horizontal internal pressure gradient in the x-direction becomes

$$\frac{\partial p_{int}}{\partial x} = gD \int_{\sigma}^{0} \left(\frac{\partial \rho}{\partial x^{*}} - \frac{\sigma}{D} \frac{\partial \rho}{\partial \sigma} \frac{\partial D}{\partial x^{*}} \right) d\sigma$$

- Near steep topography, the two terms of the integrand can be large, comparable in magnitude, and opposite of sign
- Very sensitive to truncation errors

The story

- Haney [1991] introduced the concept of *hydrostatic consistency* for ocean models
- For a hydrostatically consistent scheme

$$\left| \frac{\sigma}{\delta\sigma} \frac{\delta D}{D} \right| < 1 \tag{1}$$

- When $\delta\sigma$ approaches zero, (1) will be violated closer to the bottom boundary

The story: Mellor et al. [1994]

- Condition of hydrostatic consistency is quite restrictive [Mellor et al. 1994]
- The discretisation error is given by

$$E\left(\frac{\delta_x b}{\delta x}\right) = \frac{D}{4}\frac{\delta_x D}{\delta x}\left(\frac{\partial^2 b}{\partial z^2}\right)\left[(\delta\sigma)^2 - \sigma^2\left(\frac{\delta_x D}{D}\right)^2\right]$$

- The error decreases if $\delta\sigma$ and $\frac{\delta_x D}{D}$ goes to zero
- Hydrostatic consistency is accordingly not a meaningful concept [Berntsen and Oey, 2010]

The story: Consolation

Mellor et al. [1994] concluded: The error is not numerically divergent, and therefore *not of great concern*

The story: SEFK and SESK

Beckmann and Haidvogel [1993]

- Treated the seamount case
- Experienced growing errors

Mellor et al. [1998] – Two kinds of errors

- Sigma Errors of the First Kind (SEFK): 2D errors, will die out
- Sigma Errors of the Second Kind (SESK): Vorticity 3D errors, will not die out

Possible remedies

- Subtract a background stratification, $\rho(z)$
  [Gary, 1973]
- Add viscosity
  [Mellor, 1998]
- Smooth the topography
  [Barnier et al., 1998]

Possible remedies, cont.

- Interpolate back to z-levels
  [Stelling and van Kester, 1994]
- Higher order schemes
  [McCalpin, 1994], [Chu and Fan, 1997]
- Green's theorem approach
  [Shchepetkin and McWilliams, 2003]
- Rotate grid 45 degrees and weight
  [Thiem and Berntsen, 2006]

It is still desirable to find a solution to this problem

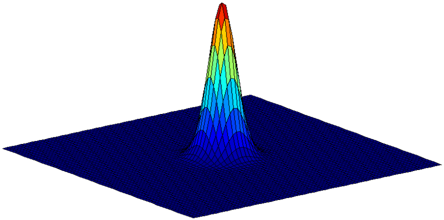
The new approaches

The following will be investigated:

- Finite volume approach for pressure computation
- Higher order schemes
- Expanded computational stencils
- Weighting of different methods

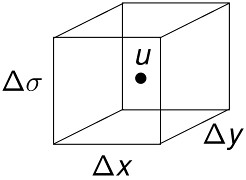
The model and the test case

- Linearised version of the Princeton Ocean Model
- The seamount test case
- No forcing, horizontal isopycnals –
  true solution is zero velocities



The finite volume approach

- Estimating pressure forces over fixed volumes



- Discretising the following integral

$$\frac{\partial p}{\partial x} = g \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\int_{\sigma}^{0} \left(D \frac{\partial \rho}{\partial x} + \frac{\partial D}{\partial x} \rho \right) d\sigma + \sigma \rho \frac{\partial D}{\partial x} \right] dxdy$$

Method A

- The standard 2nd order POM method can be interpreted as the simplest in a family of finite volume methods

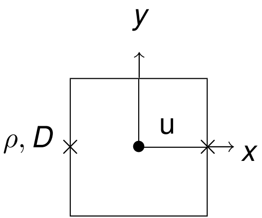
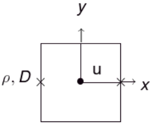
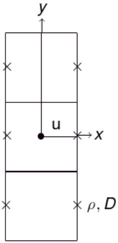


Figure: Points ρ, D ($\times$) and u ($\bullet$) at a σ-level used in the standard 2nd order POM method, Method A.

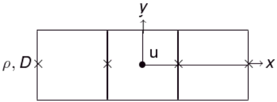
The new methods



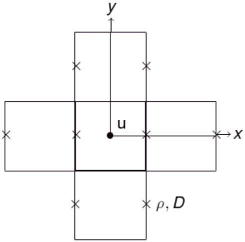
Method A



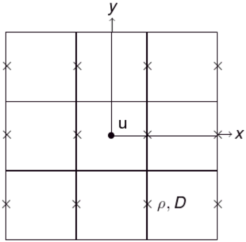
Method B



Method C



Mehtod D

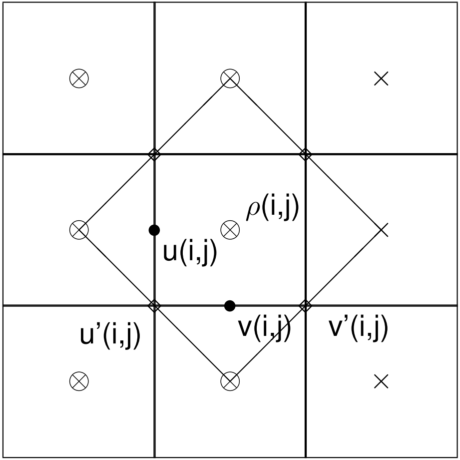


Method E

The rotated grid

- Consistent Methods A, C, and D were successfully created
- For the stencils of the Methods B and E, a new approach is needed: Rotated grid [Thiem and Berntsen, 2006]

The rotated grid



The rotated grid: A connection

- Using the stencil of Method B –, it can be shown that the rotated approach is a 2nd order finite volume method
- The rotated approach can be weighted with the 2nd order POM method, with a *fixed* weight, to reduce the errors

- The optimal fixed weight can be found by experiment
- The same can be done for the stencil of Method E

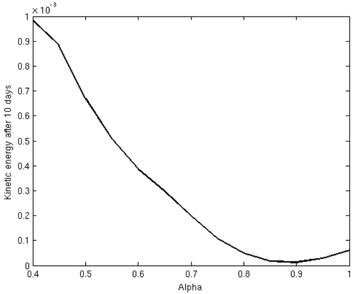


Figure: The kinetic energy after 10 days as a function of α for the Method B_α experiments.

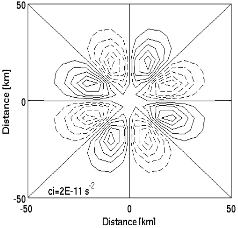
Method B_α and Method E_α

- The two resulting methods, defined by an optimal fixed weight, are called Method B_α and Method E_α
- End up with *five* consistent methods

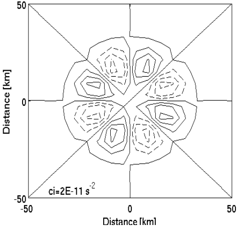
Studying the SESK

- SESK (Sigma Errors of the Second Kind) are related to vorticity [Mellor et al. 1998]
- Initial SESK closely connected with the later errors
- The vertically integrated initial acceleration of the vorticity is studied

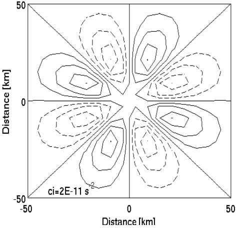
The initial vorticity error, different methods



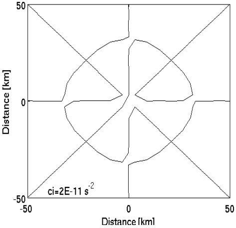
(a) Method A



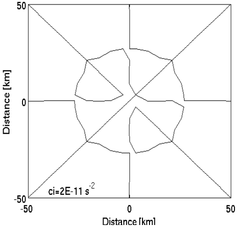
(b) Method B_α



(c) Method C



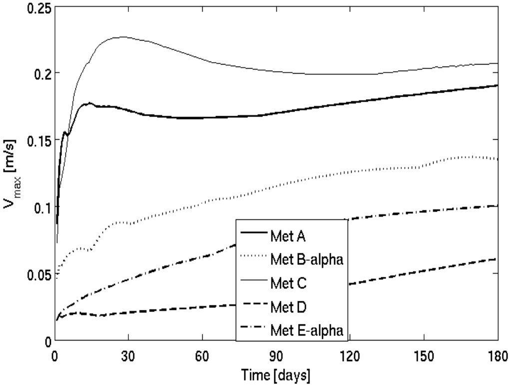
(d) Method D



(e) Method E_α

The performance of the methods

The maximum velocity, 180 days run



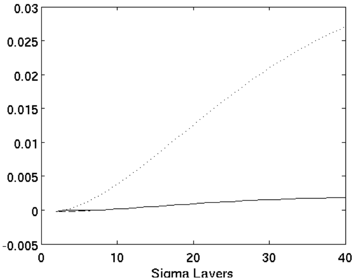
The optimal weighting

- Possible to find a field of optimal weights?
- Two estimates of the internal pressure gradient, interpolated to give the exact result in each point
- Weights $\alpha_x(x, y, \sigma)$ and $\alpha_y(x, y, \sigma)$ fixed in time
- For our case, the result should be zero

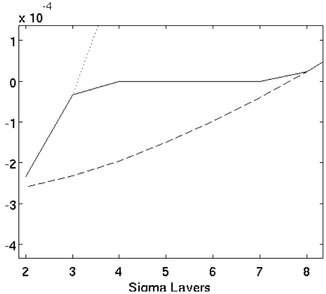
$$\alpha_x(x, y, \sigma) \cdot \left(\frac{\partial \rho}{\partial x}\right)_1 + [1 - \alpha_x(x, y, \sigma)] \cdot \left(\frac{\partial \rho}{\partial x}\right)_2 \quad = \quad 0$$

Results: The optimal weighting

- For interpolation, we need estimates of oppsite signs
- The optimal weighting not complete
- Leads to discontinuities, and instability



(a)



(b)

The way forward

- New pair of methods
- Optimal weighting also time-dependent?
- Weighting in the vertical

Concluding remarks

- The σ-coordinate system is still being used
- Many advantages to the terrain-following approach
- The severity of the internal pressure gradient problem diminishes with higher resolution

Thank you for your attention!