Really TVD advection schemes for shelf seas

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Introduction

Definition

✔ A scheme is said to be **T**otal **V**ariation **D**iminishing (**TVD**) when it ensures that

$$TV^{n+1}(y) \leq TV^{n}(y)$$

✔ The Total Variation $TV^n(y)$ of a dicrete solution for a variable y is defined as

$$TV^n(y) = \sum_i \left| y^n_{i+1} - y^n_i \right|$$

where y^n_i is the variable at time $t^n = n \triangle t$ and location $x_i = i \triangle x$

Properties

- ✔ Monotonicity preserving
  - ▷ no new extremum can be created
  - ▷ the value of a local maximum (minimum) does not increase (decrease)
- ✔ Second or higher order of accuracy in smooth part of the solution
- ✔ Solution free from spurious oscillations around discontinuities
- ✔ Do not produce too much diffusion

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The question

TVD schemes developed for

$$\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x} = 0$$

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$$\frac{\partial y}{\partial t} + u \frac{\partial y}{\partial x} = 0$$

$$\frac{\partial H y}{\partial t} + \frac{\partial H u y}{\partial x} = 0$$

continuity

- conservative form
- easier to compute mass budget
- should not produce
  - neither unreal extremum
  - nor negative value

The question

TVD schemes developed for

$$\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x} = 0$$

$$\Downarrow$$

$$\frac{\partial Hy}{\partial t} + \frac{\partial Huy}{\partial x} = 0$$

— continuity

- conservative form
- easier to compute mass budget
- should not produce
  - neither unreal extremum
  - nor negative value

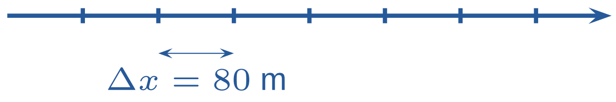
$$\Downarrow$$

TVD ?

Non TVD behaviour

Test case setup

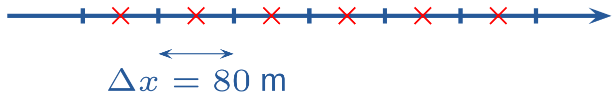
$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial Hu}{\partial x} = 0 \\ \dfrac{\partial Hy}{\partial t} + \dfrac{\partial Huy}{\partial x} = 0 \end{cases}$$



- ✔ fully explicit discretization
- ✔ regular staggered grid ($\triangle x = 80m$)

Test case setup

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial Hu}{\partial x} = 0 \\[2ex] \dfrac{\partial Hy}{\partial t} + \dfrac{\partial Huy}{\partial x} = 0 \end{cases}$$



- ✔ fully explicit discretization
- ✔ regular staggered grid ($\triangle x = 80m$)
- ✔ H and y defined at the centers of grid boxes

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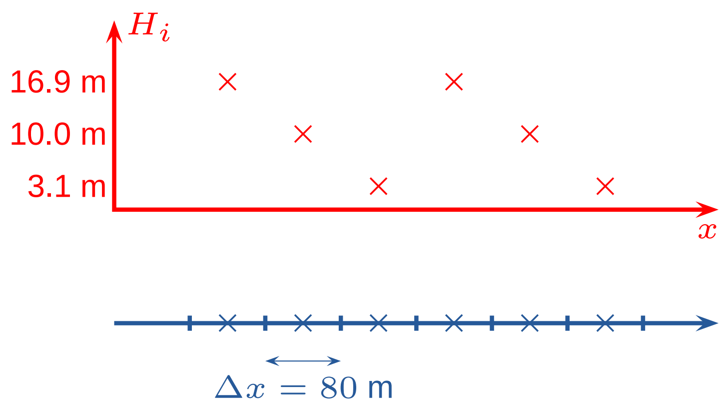

- ✔ fully explicit discretization
- ✔ regular staggered grid ($\triangle x = 80m$)
- ✔ H and y defined at the centers of grid boxes
- ✔ periodically rippled sea-floor

Test case setup

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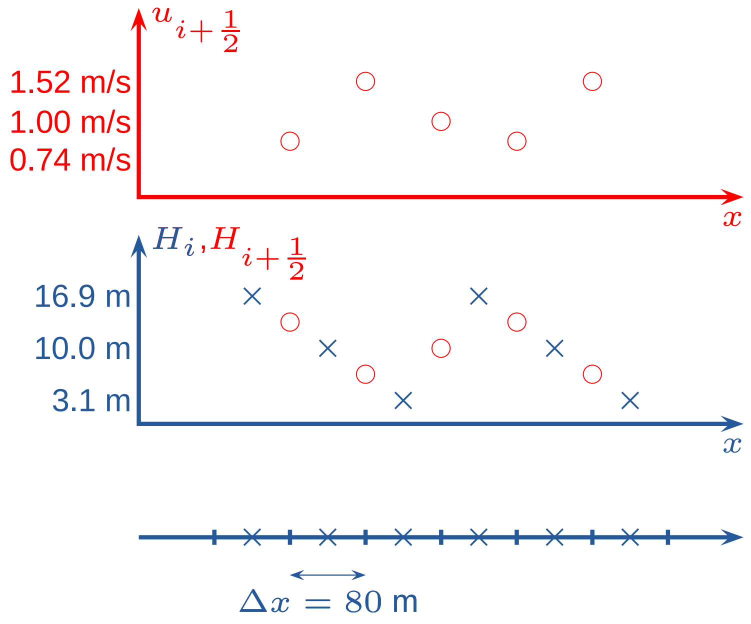

- ✔ fully explicit discretization
- ✔ regular staggered grid ($\triangle x = 80m$)
- ✔ H and y defined at the centers of grid boxes
- ✔ periodically rippled sea-floor
- ✔ Hu defined at the interfaces ($Hu = 10\ m^2/s$)

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Numerical scheme

Lax-Wendroff scheme equipped with a limiter $\phi_{i+1/2}$

$$\tilde{y}^n_{i+1/2} = y^n_i + \frac{\phi_{i+1/2}}{2}\left(1 - \nu_{i+1/2}\right)\left(y^n_{i+1} - y^n_i\right)$$

where

$$\begin{cases} \nu_{i+1/2} = \dfrac{u_{i+1/2}\Delta t}{\Delta x} \\ u_{i+1/2} = \dfrac{(Hu)^n_{i+1/2}}{H^n_{i+1/2}} \\ H^n_{i+1/2} = \dfrac{H^n_i + H^n_{i+1}}{2} \end{cases}$$

Flux limiter

In the linear case

✔ $\phi_{i+1/2}$ is usually chosen as

$$\phi_{i+1/2} = \phi\left(r_{i+1/2}\right) \quad \text{where} \quad r_{i+1/2} = \frac{y_i^n - y_{i-1}^n}{y_{i+1}^n - y_i^n}$$

Flux limiter

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- ✔ the limited Lax-Wendroff scheme is TVD if

$$\phi\left(r_{i+1/2}\right) \leq \min\left(2r_{i+1/2}, 2\right)$$

provided that $0 \leq \nu \leq 1$

Flux limiter

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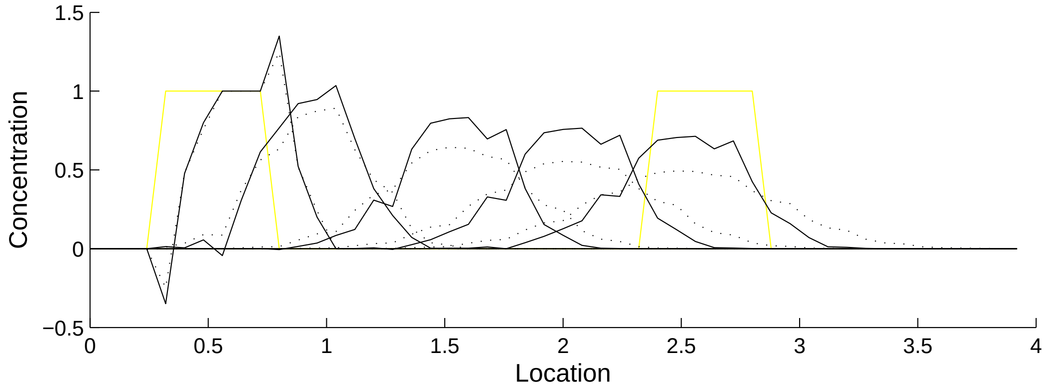
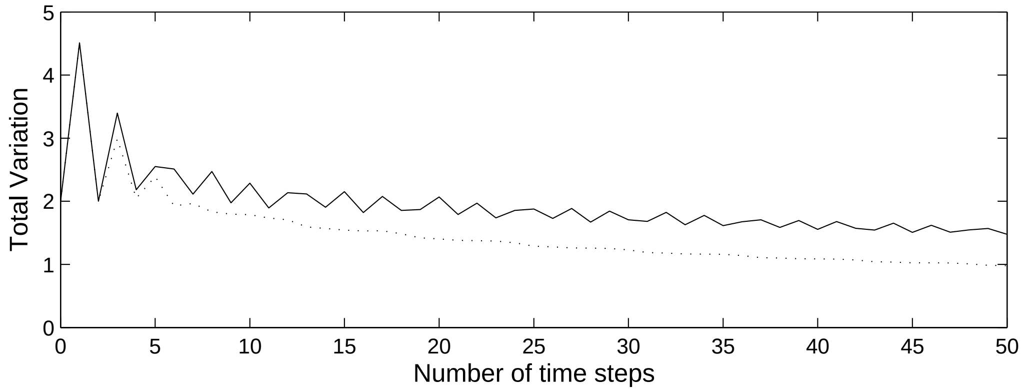
provided that $0 \leq \nu \leq 1$

✔ Superbee limiter

$$\phi_{Superbee}(r) = \max\left\{0, \min(2r, 1), \min(r, 2)\right\}$$

Results





Advection of a square initial distribution on the rippled sea-floor (CFL=0.76) using a first order upwind scheme (dotted line) and a supposedly TVD upwind/Lax-Wendroff scheme with superbee limiter (solid line), the light curve representing the analytical solution. Spurious maxima are apparent in the snapshots of the concentration field (upper figure) computed with the supposedly TVD upwind/Lax-Wendroff scheme. The total variation shows oscillations with both advection schemes (lower figure).

Solutions

Analysis

Harten (1983): A scheme is TVD if it can be written in the form

$$y_{i+1}^{n+1} = y_i^n - \mathcal{C}^-_{i-\frac{1}{2}}\left(y_i^n - y_{i-1}^n\right) + \mathcal{C}^+_{i+\frac{1}{2}}\left(y_{i+1}^n - y_i^n\right)$$

with $\mathcal{C}^{\pm}_{i+\frac{1}{2}} \geq 0$ and $\mathcal{C}^+_{i+\frac{1}{2}} + \mathcal{C}^-_{i+\frac{1}{2}} \leq 1$

Linear: $$0 \leq \nu\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}(1-\nu)\right] + \frac{\nu}{2}(1-\nu)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

- provided that $0 \leq \nu \leq 1$
- Classical TVD range
- Superbee limiter

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Linear:

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Depth-integrated:

$$0 \leq \nu^+_{i-\frac{1}{2}} \left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}\left(1 - \nu_{i-\frac{1}{2}}\right)\right] + \frac{\nu^-_{i+\frac{1}{2}}}{2}\left(1 - \nu_{i+\frac{1}{2}}\right)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

where $\nu^-_{i+\frac{1}{2}} = \frac{(Hu)^n_{i+\frac{1}{2}}}{H_i^{n+1}\triangle x}$, $\nu^+_{i-\frac{1}{2}} = \frac{(Hu)^n_{i-\frac{1}{2}}}{H_i^{n+1}\triangle x}$

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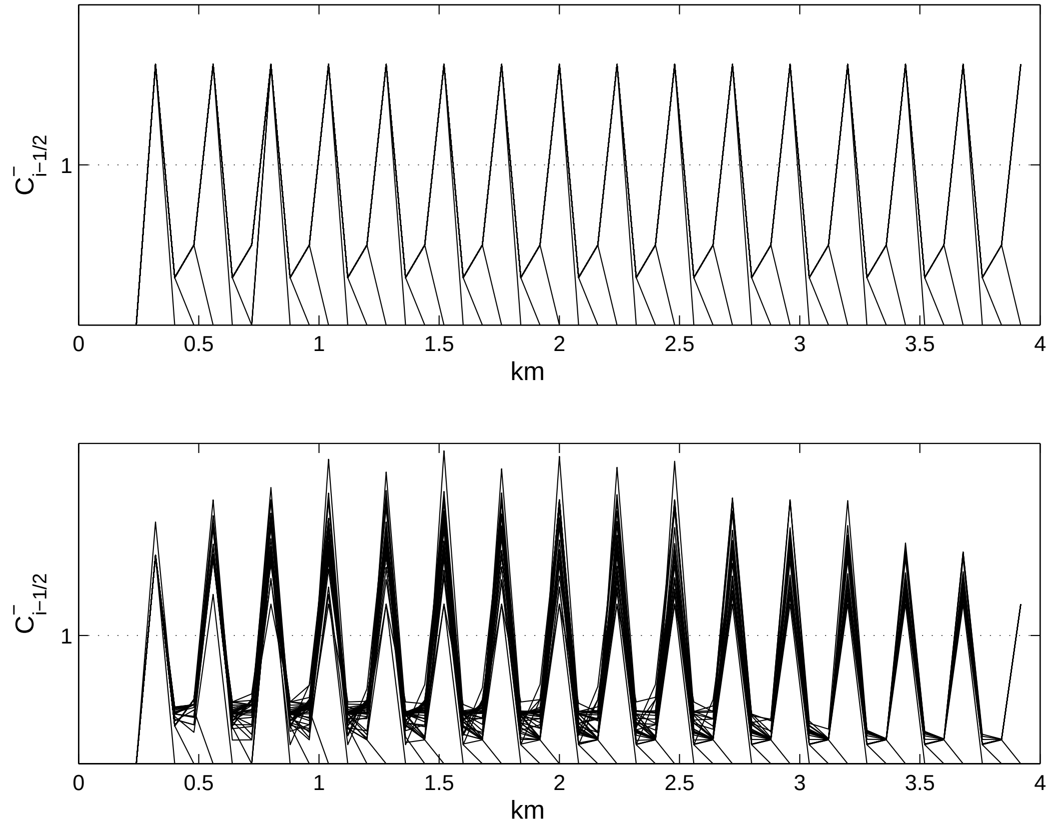
Analysis



Snapshots of the coefficient $\mathcal{C}^-_{i-1/2}$ using a first order upwind scheme (upper figure) and an upwind/Lax-Wendroff scheme with superbee limiter (lower figure). Since this coefficient is occasionally larger than unity, the TVD behavior of the schemes cannot be guarantied using Harten's theorem.

Analysis

Harten (1983): A scheme is TVD if it can be written in the form

$$y_{i+1}^{n+1} = y_i^n - \mathcal{C}^-_{i-\frac{1}{2}}\left(y_i^n - y_{i-1}^n\right) + \mathcal{C}^+_{i+\frac{1}{2}}\left(y_{i+1}^n - y_i^n\right)$$

with $\mathcal{C}^{\pm}_{i+\frac{1}{2}} \geq 0$ and $\mathcal{C}^+_{i+\frac{1}{2}} + \mathcal{C}^-_{i+\frac{1}{2}} \leq 1$

Linear: $$0 \leq \nu\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}(1-\nu)\right] + \frac{\nu}{2}(1-\nu)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

- provided that $0 \leq \nu \leq 1$
- Classical TVD range
- Superbee limiter

Depth-integrated: $$0 \leq \nu^+_{i-\frac{1}{2}}\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}\left(1-\nu_{i-\frac{1}{2}}\right)\right] + \frac{\nu^-_{i+\frac{1}{2}}}{2}\left(1-\nu_{i+\frac{1}{2}}\right)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

where $$\nu^-_{i+\frac{1}{2}} = \frac{(Hu)^n_{i+\frac{1}{2}}}{H_i^{n+1}\triangle x}, \quad \nu^+_{i-\frac{1}{2}} = \frac{(Hu)^n_{i-\frac{1}{2}}}{H_i^{n+1}\triangle x}$$

Analysis

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$$y_{i+1}^{n+1} = y_i^n - \mathcal{C}^-_{i-\frac{1}{2}}\left(y_i^n - y_{i-1}^n\right) + \mathcal{C}^+_{i+\frac{1}{2}}\left(y_{i+1}^n - y_i^n\right)$$

with $\mathcal{C}^\pm_{i+\frac{1}{2}} \geq 0$ and $\mathcal{C}^+_{i+\frac{1}{2}} + \mathcal{C}^-_{i+\frac{1}{2}} \leq 1$

Linear:

$$0 \leq \nu\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}(1-\nu)\right] + \frac{\nu}{2}(1-\nu)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

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Depth-integrated:

$$0 \leq \nu^+_{i-\frac{1}{2}}\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}\left(1 - \nu_{i-\frac{1}{2}}\right)\right] + \frac{\nu^-_{i+\frac{1}{2}}}{2}\left(1 - \nu_{i+\frac{1}{2}}\right)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

where $\nu^-_{i+\frac{1}{2}} = \frac{(Hu)^n_{i+\frac{1}{2}}}{H_i^{n+1}\triangle x}$, $\nu^+_{i-\frac{1}{2}} = \frac{(Hu)^n_{i-\frac{1}{2}}}{H_i^{n+1}\triangle x}$

Analysis

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$$y_{i+1}^{n+1} = y_i^n - \mathcal{C}_{i-\frac{1}{2}}^- \left(y_i^n - y_{i-1}^n\right) + \mathcal{C}_{i+\frac{1}{2}}^+ \left(y_{i+1}^n - y_i^n\right)$$

with $\mathcal{C}_{i+\frac{1}{2}}^{\pm} \geq 0$ and $\mathcal{C}_{i+\frac{1}{2}}^{+} + \mathcal{C}_{i+\frac{1}{2}}^{-} \leq 1$

Linear: $$0 \leq \nu \left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}(1-\nu)\right] + \frac{\nu}{2}(1-\nu)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

Modify the scheme

- provided that $0 \leq \nu \leq 1$
- Classical TVD range
- Superbee limiter

Depth-integrated: $$0 \leq \nu_{i-\frac{1}{2}}^+ \left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}\left(1 - \nu_{i-\frac{1}{2}}\right)\right] + \frac{\nu_{i+\frac{1}{2}}^-}{2}\left(1 - \nu_{i+\frac{1}{2}}\right)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

where $$\nu_{i+\frac{1}{2}}^- = \frac{(Hu)_{i+\frac{1}{2}}^n}{H_i^{n+1}\triangle x}, \quad \nu_{i-\frac{1}{2}}^+ = \frac{(Hu)_{i-\frac{1}{2}}^n}{H_i^{n+1}\triangle x}$$

Analysis

Harten (1983): A scheme is TVD if it can be written in the form

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with $\mathcal{C}^{\pm}_{i+\frac{1}{2}} \geq 0$ and $\mathcal{C}^+_{i+\frac{1}{2}} + \mathcal{C}^-_{i+\frac{1}{2}} \leq 1$

Linear: $0 \leq \nu\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}(1-\nu)\right] + \frac{\nu}{2}(1-\nu)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$

- provided that $0 \leq \nu \leq 1$
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Modify the scheme

Depth-integrated: $0 \leq \nu^+_{i-\frac{1}{2}}\left[1 - \frac{\phi_{i-\frac{1}{2}}}{2}\left(1 - \nu_{i-\frac{1}{2}}\right)\right] + \frac{\nu^-_{i+\frac{1}{2}}}{2}\left(1 - \nu_{i+\frac{1}{2}}\right)\frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$

Adapt the TVD range and limiters

Modified scheme

✔ Upwind scheme is TVD if $0 \leq \mathcal{C}^{-}_{i-\frac{1}{2}} = \nu^{+}_{i-\frac{1}{2}} \leq 1$

✔ Lax-Wendroff scheme is TVD if:

▷ use $\nu^{\pm}_{i-\frac{1}{2}}$ as upwind rates for the LW fluxes

▷ to ensure unicity of the flux at the interface, choose

$$\nu_{i-1/2} = \max\left(\nu^{+}_{i-1/2}, \nu^{-}_{i-1/2}\right)$$

▷ ensure that

$$0 \leq \nu_{i-1/2} \leq 0.75$$

Adapt TVD range

✔ Classical: $$0 \leq \mathcal{C}^{-}_{i-\frac{1}{2}} = \nu \left[1 - \frac{\phi_{i-\frac{1}{2}}}{2} (1-\nu) \right] + \frac{\nu}{2} (1-\nu) \frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

⇓ Using Sweby's method (1984) and provided that $0 \leq \nu \leq 1$

$$\begin{cases} 0 \leq \phi_{i+1/2} \leq \min \{2, 2r_{i+1/2}\}, & \text{for } r_{i+1/2} > 0 \\ \phi_{i+1/2} = 0, & \text{for } r_{i+1/2} \leq 0 \end{cases}$$

Adapt TVD range

✔ Classical: $0 \leq \mathcal{C}^{-}_{i-\frac{1}{2}} = \nu \left[1 - \frac{\phi_{i-\frac{1}{2}}}{2} (1-\nu) \right] + \frac{\nu}{2} (1-\nu) \frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$

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$$\begin{cases} 0 \leq \phi_{i+1/2} \leq \min \left\{ 2, 2r_{i+1/2} \right\}, & \text{for } r_{i+1/2} > 0 \\ \phi_{i+1/2} = 0, & \text{for } r_{i+1/2} \leq 0 \end{cases}$$

✔ Depth-integrated case:

$$0 \leq \mathcal{C}^{-}_{i-\frac{1}{2}} = \nu^{+}_{i-\frac{1}{2}} \left[1 - \frac{\phi_{i-\frac{1}{2}}}{2} \left(1 - \nu_{i-\frac{1}{2}} \right) \right] + \frac{\nu^{-}_{i+\frac{1}{2}}}{2} \left(1 - \nu_{i+\frac{1}{2}} \right) \frac{\phi_{i+\frac{1}{2}}}{r_{i+\frac{1}{2}}} \leq 1$$

⇓ Using Sweby's method (1984) and provided that $0 \leq \nu, \nu^{+}, \nu^{-} \leq 1$

$$\begin{cases} 0 \leq \phi_{i+1/2} \leq \min \left\{ \frac{2}{1-\nu_{i+1/2}}, \frac{2r_{i+1/2}(1-\nu^{+}_{i-1/2})}{\nu^{-}_{i+1/2}\left(1-\nu_{i+1/2}\right)} \right\}, & \text{for } r_{i+1/2} > 0 \\ \phi_{i+1/2} = 0, & \text{for } r_{i+1/2} \leq 0 \end{cases}$$

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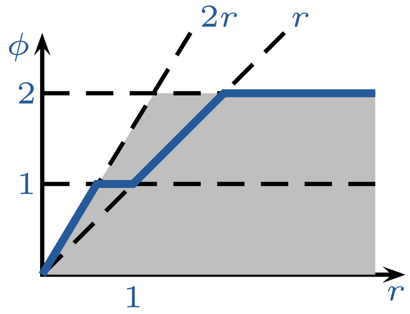
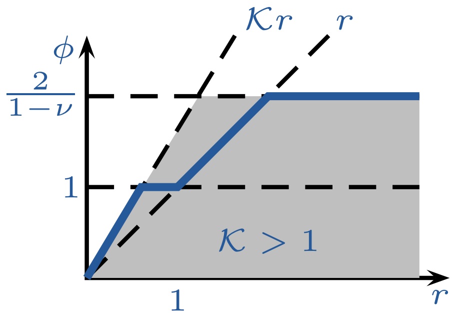
Adapt TVD range

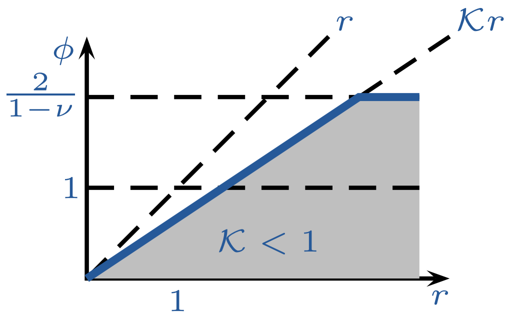
✔ Classical:



$$\phi\text{sup}(r) = \max\{0, \min(2r, 1), \min(r, 2)\}$$

✔ Depth-integrated case:

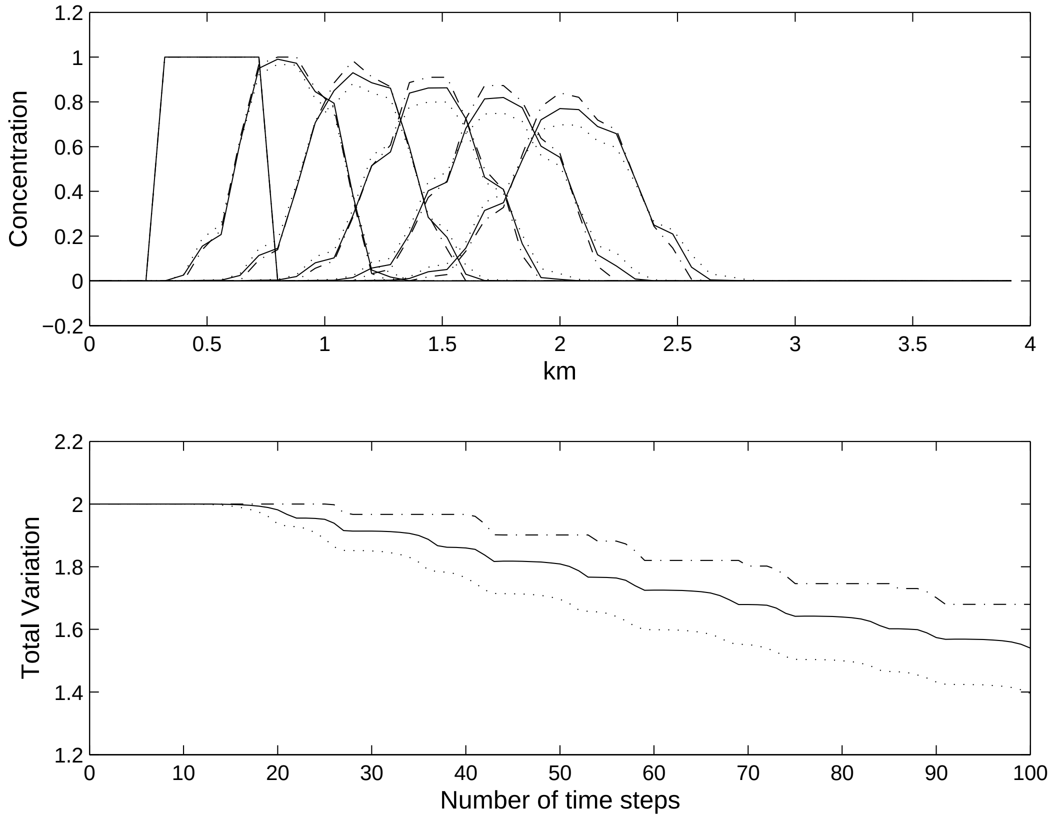




$$\mathcal{K} = \frac{2(1-\nu^{+}_{i-1/2})}{\nu^{-}_{i+1/2}\left(1-\nu_{i+1/2}\right)}$$

$$\begin{cases} \mathcal{K} > 1 : \ \phi\text{sup}(r_{i+\frac{1}{2}}) = \max\left\{0, \min\left(r, \frac{2}{1-\nu}\right), \min\left(\mathcal{K}r, 1\right)\right\} \\ \mathcal{K} < 1 : \ \phi\text{sup}(r_{i+\frac{1}{2}}) = \max\left\{0, \min\left(\mathcal{K}r, \frac{2}{1-\nu}\right)\right\} \end{cases}$$

Results



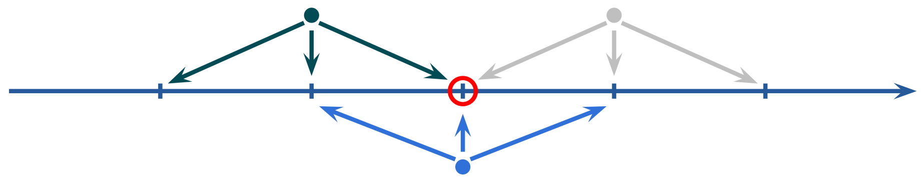
Advection of a square initial distribution on the rippled sea-floor using the original upwind/Lax-Wendroff scheme with superbee limiter (solid line), using the modified scheme (dotted line) and using the generalized superbee limiter (dashed-dotted line). A small time step of 15 s is used to satisfy the modified CFL condition. The snapshots of the concentration field (upper figure) confirm that the modified scheme introduces the largest numerical diffusion while the generalized superbee limiter is the less diffusive scheme. The Total Variation (lower figure) associated with the three discrete solutions decreases monotonically.

Generalization

Difficulties

✔ Find the right form of coefficients $\mathcal{C}^{\pm}_{i\pm\frac{1}{2}}$ in order to use Harten's theorem

✔ 8 arrangements of signs of 3 successive values of Hu give conditions on their 3 relative limiters

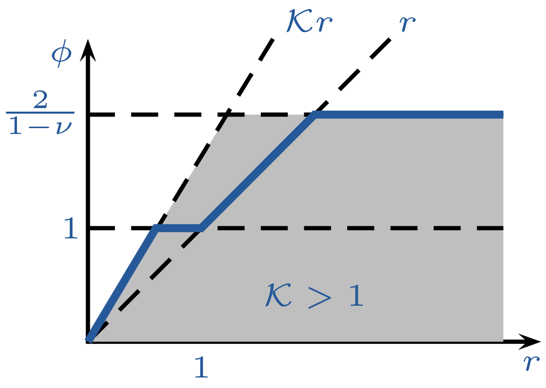


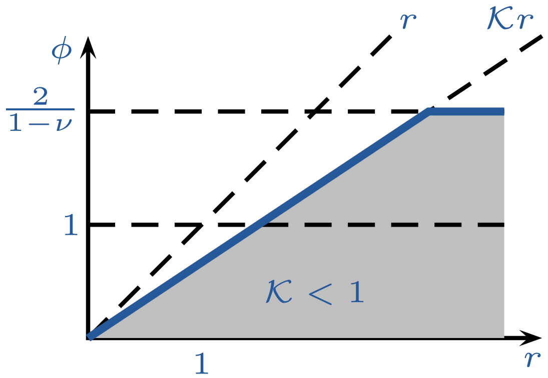
✔ to obtain the set of conditions relative to a specific interface, 32 arrangements of signs of 5 successive values of Hu have thus to be considered

General range

$$\begin{cases} 0 \leq \phi^s_{i+1/2} \leq \text{min}\left(\dfrac{2}{1-\nu_{i+1/2}}, \mathcal{K} r^{-s}_{i+1/2}\right) & \text{for } r^{-s}_{i+1/2} > 0 \\ \phi^s_{i+1/2} = 0 & \text{for } r^{-s}_{i+1/2} \leq 0 \end{cases}$$

where $s = \text{sign}\,(Hu)_{i+1/2}\,, \quad r^{\pm}_{i+1/2} = \dfrac{\Delta_{i+1/2\pm 1}y}{\Delta_{i+1/2}y}$





- $\mathcal{K} = \dfrac{2\left(1-\nu^s_{i+\frac{1}{2}-s}\right)}{\nu^{-s}_{i+\frac{1}{2}}\left(1-\nu_{i+\frac{1}{2}}\right)}$ if $(Hu)_{i+\frac{1}{2}} \cdot (Hu)_{i+\frac{1}{2}-s} \geq 0$
- $\mathcal{K} = \dfrac{2\left(1-\nu^{-s}_{i+\frac{1}{2}-s}\right)}{\nu^{-s}_{i+\frac{1}{2}}\left(1-\nu_{i+\frac{1}{2}}\right)}$ if $\begin{cases} (Hu)_{i+\frac{1}{2}} \cdot (Hu)_{i+\frac{1}{2}-s} < 0 \\ (Hu)_{i+\frac{1}{2}} \cdot (Hu)_{i+\frac{1}{2}-2s} \geq 0 \end{cases}$
- $\mathcal{K} = 0$ if $\begin{cases} (Hu)_{i+\frac{1}{2}} \cdot (Hu)_{i+\frac{1}{2}-s} < 0 \\ (Hu)_{i+\frac{1}{2}} \cdot (Hu)_{i+\frac{1}{2}-2s} < 0 \end{cases}$

Conclusion

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Conclusion

- ✔ Classical TVD schemes (linear advection) can lead to non-TVD solution for the depth-integrated transport equation
- ✔ The TVD character can be recovered if
  - ▷ a modified CFL condition that takes into account the local variation of H is satisfied
  - ▷ the TVD range is adapted to the specific depth-integrated advection equation
  - ▷ A generalized superbee limiter can then be derived
- ✔ All developments can be generalized in the case of variable velocity
- ✔ Applicable to 2D depth-integrated models or 3D models in σ-coordinates, ...