On the use of a depth-dependent barotropic mode for free surface ocean models

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Bryan 69 : First splitting & Rigid-lid approximation

Bryan 69 : Rigid-lid approximation & splitting:

- First numerical 3D ocean model.
- Primitive equations, hydrostatic, Boussinesq.
- Rigid-lid approximation = no fast waves.
- Already one splitting method enforced by the elliptic surface pressure problem.
- Mathematically true since there is no free-surface : orthogonality.
- Pb-development : very ill conditionned, assimilation, middle scale, internal waves representation, parallelization

Let the vertical average over the whole water column be indicated by an overbar,

$$\overline{(\;)} = \frac{1}{H}\int_{-H}^{0} (\;)\, dz \qquad (2.29)$$

and the deviation from a vertical average by $\hat{(\;)}$. Thus the velocity components may be expressed as

$$(u, v) = (\bar{u}, \bar{v}) + (\hat{u}, \hat{v}). \qquad (2.30)$$

The $\bar{u}$, $\bar{v}$ components may be predicted from (2.23) and (2.28). To predict $\hat{u}$, $\hat{v}$ we make use of (2.3) and (2.4) with the right-hand side of (2.24) substituted for the pressure term. The surface pressure, p_s, is temporarily set to zero:

$$u_t' + \mathscr{L}u - 2\Omega n v - mnuv/a = \frac{-mg}{\rho_0 a}\left(\int_z^0 \rho\, dz\right)_\lambda + F^\lambda \qquad (2.31)$$

$$v_t' + \mathscr{L}v + 2\Omega n u + mnuu/a = \frac{-g}{\rho_0 a}\left(\int_z^0 \rho\, dz\right)_\varphi + F^\varphi. \qquad (2.32)$$

u' and v' differ from u and v due to the neglect of that part of the pressure gradient force which depends on the surface pressure. To determine $\hat{u}$ and $\hat{v}$ we set

$$(\hat{u}, \hat{v}) = (u' - \bar{u}', v' - \bar{v}'). \qquad (2.33)$$

In the determination of $\hat{u}$ and $\hat{v}$ the error due to the neglect of surface pressure in u' and v' is of no consequence, since that error is independent of z and is therefore eliminated by subtracting out $\bar{u}'$ and $\bar{v}'$.

K.Bryan : A Numerical Method for the Study of the Circulation of the World Ocean, JCP, 1969

Free surface model and splitting

1. Bryan 69 : Rigid-lid approximation & splitting
2. R.Madala 77, P.D.Killworth et al. 91: Free surface model and splitting

- Free surface equation : $\partial_t \eta + \nabla UH = 0$
- Pb n°1 : No more Poisson like solver but one more equation (U needed).
- Pb n°2 : Introduction of surface gravity waves (very fast).
- Solution : In linear study the vertical average of velocity seems to be orthogonal to the rest of the flow (barotropic aka external mode) $\Longrightarrow$ Compute apart the barotropic (external) system (splitting method).
- Pb n°3: Mathematically not true, since there is a free-surface : false orthogonality.
- Pb n°4: Not possible to exprime the exact barotropic (averaged) part of the non linear term : $u\nabla u$ .... and friction terms ??
- Pb-development : stability, physics of internal waves, filter, explicit (subcycling) or implicit ?

Numerical stability study and filter

1. Bryan 69 : Rigid-lid approximation & splitting
2. R.Madala 77, P.D.Killworth et al. 91: Free surface model and splitting
3. R.Higdon et al. 96, V.M. Kamenkovich et al. 08, and Shchepetkin et al. 05 : Numerical stability study and filter
   - Numerical stability analysis on simplified cases of the barotropic splitting using the rigid-lid approximation (external mode).$\Longrightarrow$ Need of a filter in the non-linear case. (R.Higdon et al. 96, V.M. Kamenkovich et al. 98)
   - Conservation of constant trough the splitting method scheme $\Longrightarrow$ modification of the tracer advection velocity. (Shchepetkin 03)
   - Pb : Large damping

Splitting : Barotropic mode vs External mode, improvment ?

1. Bryan 69 : First splitting & Rigid-lid approximation
2. R.Madala 77, P.D.Killworth et al. 91: Free surface model and splitting
3. R.Higdon et al. 96, V.M. Kamenkovich et al. 98, and Shchepetkin et al. 05 : Numerical stability study and filter
4. And now :

   Keeping the orghogonality in the linear case $\Longrightarrow$ reduce the need and effects of the filter in the non-linear case ??

A few notions you have to keep in mind

Rigid-Lid / Free-Surface

Internal gravity waves

Orthogonality

Linear theory

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Linearized system

Linearized primitive equations :

$$\begin{cases} \partial_t u + \frac{1}{\rho_0}\partial_x p = 0 \\ \partial_x u + \partial_z w = 0 \\ \rho = -\frac{1}{g}\partial_z p \\ \partial_t \rho + w\partial_z \bar{\rho} = 0 \end{cases} \Longrightarrow \begin{cases} \partial_t u + \frac{1}{\rho_0}\partial_x p = 0 \\ \partial_x u - \Lambda(\frac{1}{\rho_0}\partial_t p) = 0 \\ \rho = -\frac{1}{g}\partial_z p \\ w = -N^{-2}\partial_{zt} p \end{cases}$$

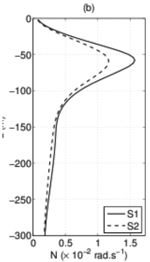
Boundary conditions

$$\begin{cases} w(x,-H,t) = 0 & \textit{Flat bottom} \\ w(x,0,t) = \partial_t \eta \end{cases}$$

Kundu and Cohen, 1990

Notations:

- $\Lambda = \partial_z(N^{-2}\partial_z)$
- The Brunt-Vaisala frequency : $N^{-2} = -\frac{\rho_0}{g}(\partial_z\bar{\rho})^{-1}$



Definition of normal modes

Variables projection:

- 2D projected variables :

$$\forall n : \begin{cases} u_n(x,t) = \langle u, M_n \rangle \\ p_n(x,t) = \langle p, M_n \rangle \end{cases}$$

- $$\begin{cases} u = \sum_{n=1}^{\infty} u_n M_n \\ p = \sum_{n=1}^{\infty} p_n M_n \\ \rho = -g \sum_{n=1}^{\infty} p_n \partial_z M_n \end{cases}$$

Definition of vertical modes M_n:

- Scalar product : $\langle f, g \rangle = \frac{1}{H} \int_{-H}^{0} f(z) g(z) dz$
- $$\begin{cases} \Lambda M_n = -c_n^{-2} M_n \\ \langle M_n, M_m \rangle = \delta_{mn} \end{cases}$$
- Boundary conditions :

$$\begin{cases} \frac{dM_n}{dz}(-H) = 0 \\ M_n(0) = -gN^{-2} \frac{dM_n}{dz}(0) \end{cases}$$

Equations projection: decoupled hyperbolic system

We get an hyperbolic system on each independent characteristic variable :

$$\forall n : \begin{cases} \partial_t y^+ + c_n \partial_x y^+ = 0 \\ \partial_t y^- - c_n \partial_x y^- = 0 \end{cases}$$ with : $y_\pm = \frac{1}{2}(u_n \pm \frac{1}{c_n \rho_0} p_n)$

The primitive system can be entirely decomposed in a sum of independent internal waves ($y_n^\pm$), evolving with their own constant speed ($\pm c_n$)

Kundu and Cohen, 1990

Magnitudes and splitting motivations

Under the approximation of a constant stratification : $\partial_z N = 0$

Depth dependent mode definition

- Top boundary condition on modes : $M_n(0) = -gN^{-2}\frac{dM_n}{dz}(0)$
- Barotropic mode : $\partial_z M_0 \neq 0, c_0 \simeq \sqrt{gH}$
- Baroclinic modes : $\forall k \geq 1, c_k \simeq \frac{NH}{k\pi}$

Rigid-lid approximation : $\partial_t \eta = 0$

- Top boundary condition on modes : $\frac{dM_n}{dz}(0) = 0$
- External mode : $\partial_z M_0 = 0, c_0 = \infty$
- Baroclinic modes : $\forall k \geq 1, c_k = \frac{NH}{k\pi}$

Problem : $CFL_0 = \frac{c_0 \Delta T}{dx} \gg 1 \Longrightarrow$ Very strong constraint

A solution : Computing apart the 2D barotropic mode containing fastest terms

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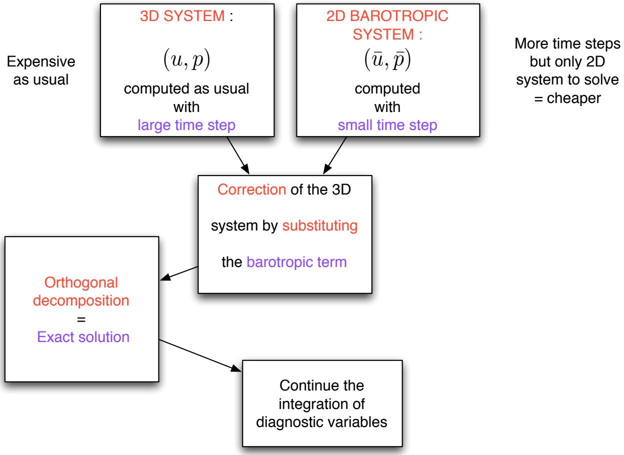
The splitting method principle



Depth dependent / independent splitting

Depth dependent barotropic splitting

$$\bar{u} = u_0 = \frac{1}{H}\int_{-H}^{0} uM_0$$

$$\bar{p} = p_0 = \frac{1}{H}\int_{-H}^{0} pM_0$$

System :

$$\begin{cases} \partial_t u_0 + \dfrac{1}{\rho_0}\partial_x p_0 = 0 \\ \partial_t p_0 + c_0^2 \rho_0 \partial_x u_0 = 0 \end{cases}$$

$c_0 \simeq \sqrt{gH}$

Correction :

$$u = u^{3d} + [u_0 - \langle u^{3d}, M_0\rangle] M_0(z)$$
$$p = p^{3d} + [p_0 - \langle p^{3d}, M_0\rangle] M_0(z)$$

and as diagnosis : $\rho = -g \sum_n p_n \partial_z M_n$

External mode (depth independent)splitting

$$\bar{u} = u_a = \frac{1}{H}\int_{-H}^{0} u$$

$$\bar{p} = \rho_0 g \eta$$

System :

$$\begin{cases} \partial_t u_a + g\partial_x \eta = -\displaystyle\int_{-H}^{0}(p - \rho_0 g\eta) \\ \partial_t \eta + \partial_x H u_a = 0 \end{cases}$$

$c_0^* = \sqrt{gH}$

Correction :

$$u = u^{3d} + [U - \langle u^{3d}, 1\rangle]$$
$$p^{3d}(0) = \rho_0 g \eta$$

no correction of the density !

Stability analysis and correction of the external mode splitting probable instability

The depth dependent barotropic mode splitting is stable by construction.

And the external mode splitting ? Why it could be unstable ?

In the external mode splitting some components of the true barotropic flow (u_0, p_0) are still integrated with the baroclinic time step. *(Killworth et al., 1991)*

$\Rightarrow$ Sources of instability

Analytical stability analysis

- Instability already proved on simplified cases (two layers) : *(Higdon and Bennett, 1996 ; Kamenkovic and Nechaev, 2008)*
- Detailed stability analysis still in progress.

Correction of this potential instability

- A filter on the 2D system or an additional diffusivity term is required *(Shchepetkin, 2005)*, even in the linear case.
- Implement a depth-dependent barotropic splitting !

Implementation of a depth dependent splitting

To easily implement the splitting on a code with an explicit density equation and a diagnosis pressure variable we have to rewrite the 2d system as a density/velocity terms !

$$\begin{cases} \partial_t u_0 + \frac{g}{\lambda\rho_0}\partial_x\left[+\frac{\rho_0}{gH}M_b(0)\eta - \bar{\rho}\right] = 0 \\ \partial_t\bar{\rho} + \frac{\rho_0}{g}\partial_x[M_0(0)u_a - u_0] = 0 \\ \partial_t\eta + \partial_x H u_a = 0 \end{cases}$$

With : $\bar{\rho}(x,t) = \int_{-H}^{0}\rho(x,z,t)N^{-2}\partial_z M_0(z)dz$

The correction, then, is directly put on the velocity (as usual) and density before the diagnostic step of pressure :

$$u^{n+1,c}(z) = u^{n+1}(z) + [u_0^{n+1} - \overline{u^{n+1}}]M_0(z)$$

$$\rho^{n+1,c}(z) = \rho^{n+1}(z) + [\frac{\bar{\rho}^{n+1} - \overline{\rho^{n+1}}}{\lambda - \frac{M_0(0)}{gH}}]\partial_z M_0(z)$$

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Test Case

Barotropic analytic test case to compare the two methods :

- Initialization with the barotropic solution (potentially unstable):

$$\rho = -g * 0.10 \sin \frac{2\pi x}{Lx} \partial_z M_0$$

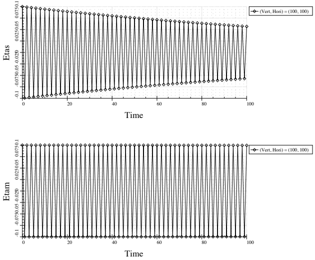
$$u = \rho_0 c_0^2 * 0.10 \sin \frac{2\pi x}{Lx} M_0(z)$$

- Analytic solution : computed with the characteristic method.
- Configuration : cyclic domain, flat bottom, $H = 4km$, $L = 15km$, $dx = 75m$, $dz = 200m$.
- Parameters : N=10^{-3} s., $CFL_0 = 0.5$, $CFL_1 = 0.2$
- Code : 2D-xz
- Coordinates : Geopotential vertical coordinates on Arakawa C-grid.
- Numerical schemes : Forward-Backward + Upwind schemes + Barotropic Splitting.

Instability and filter effects

First results :

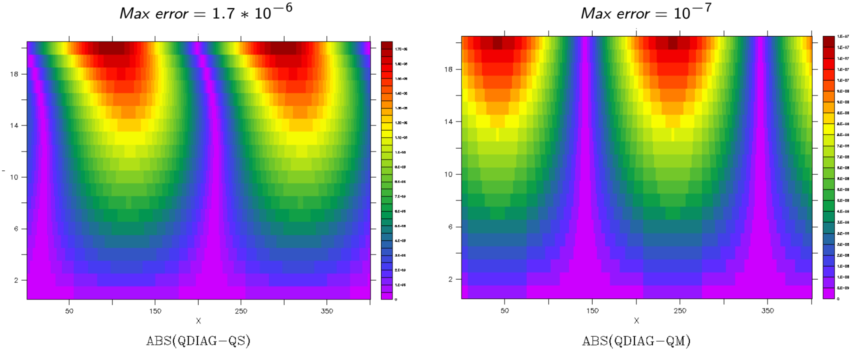
- the depth independent *(a.k.a. external mode)* splitting is unstable. We add a power-function shaped filter *(Shchepetkin, 2005)* $\Rightarrow$ Large damping of the free-surface elevation.
- the depth dependent splitting method is stable without filter and very close to the analytic solution.



Temporal series of free-surface η, top: depth independent splitting + power filter, bottom: depth dependent splitting

Density vertical shape

Influence of the depth dependent density correction on the vertical shape of the density :



depth independent splitting (external mode)

depth dependent splitting

Instantaneous plot of $|\rho_{diag} - \rho|$ *after* 2000 *time step*

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Conclusion, perspectives

CONCLUSIONS:

- The traditional external mode splitting doesn't allow density correction and is unstable even in a linear case: it needs a filter correction.
- The implementation of a depth dependent splitting in a density formulation is possible and works well in a linear case without filter.
- The work has been extended to the non-linear conservative case with similar conclusions.

PERSPECTIVES:

- Derivation of the barotropic system with a non-flat topography.
- Implementation in a realistic model and test if this study could, at the end, reduce the need and effects of filter.
- Try to re-introduce sound waves in a same way ...